The Categroy of Artificial Perceptions

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Abstract

Perception is the recognition of elements and events in the environment, usually through integration of sensory impressions. It is considered here as a broad, high-level, concept (different from the sense in which computer vision/audio research takes the concept of perception). We propose and develop premises for a formal approach to a fundamental phenomenon in AI: the diversity of artificial perceptions. A mathematical substratum is proposed as a basis for a rigorous theory of artificial perceptions. A basic mathematical category is defined. Its objects are *perceptions*, consisting of *world elements*, *connotations*, and a three-valued (*true, false, undefined*) predicative correspondence between them. Morphisms describe paths between perceptions. This structure serves as a basis for a mathematical theory. This theory provides a way of extending and systematizing certain intuitive pre-theoretical conceptions about perception, about improving and/or completing an agent's perceptual grasp, about transition between various perceptions, etc. Some example applications of the theory are analyzed.

1 Introduction

Artificial intelligent agents are, in particular, machines capable of autonomous interaction with the environment. They need to perceive their environments. Perception is the recognition of elements and events in the environment, usually through integration of sensory impressions. It is considered here as a broad, high-level, concept (different from the sense in which computer vision/audio research takes the concept of perception). Artificial Agents do not usually share the same perception. Their architecture, hardware, and sensory-motor apparatus vary, and they are conceived and programmed for different purposes by different people who build their own conceptualizations into the system, each using his own encoding. Even in cases where artificial agents do share similar architecture, hardware, and sensory-motor apparatus, they may have different transductions from the same outside world into a completely different internal reflection of that world. The various transductions may be functions of innate processes, as is often the case with human agents (some extreme examples are given in [27]).

As a simple example of the diversity of artificial perception, consider an agent and a box. The question is whether the agent perceives the box as full. Here is a partial list of possible situations:

- The agent may have been programmed, according to its purposes, to consider a box full of air as either full or not.
- What if the box is full of waste-paper, which could be disposed of. Is such a box considered full?
- What about a half full box?
- The agent may be unable to perceive the box.
- The agent may be able to perceive the box, but it has no 'idea of fullness'. (It may have a notion of 'emptiness', or not even that.)
- The agent may not perceive one world element of a box, but rather six world elements which form the sides of the box.
- The agent might have the ('expected'?) sensory-motor capabilities, but it does not bother to use them because the question is irrelevant to its current purposes.
- The agent might have (the 'expected'?) knowledge of the contents of the box, but for current purposes it is more practical not to distinguish full boxes from others.

The diversity of artificial perceptions raises the problem of how one may analyze and perform in that domain in a general, yet rigorous, manner. It is desirable to be able to state meaningful problems and results that should be applicable across the entire, diverse, spectrum of artificial perceptions.

A mathematical substratum is proposed as a basis for a rigorous theory of artificial perceptions. A basic mathematical category is defined. Its objects are *perceptions*, consisting of *world elements*, *connotations*, and a three-valued (*true, false, undefined*) predicative correspondence between them. Morphisms describe paths between perceptions.

Category theory provides tools for stating results usable across a wide spectrum of mathematical domains and objects [25, 16, 1, 4, 31, 32, 9, 6]. Defining perceptions as a domain of mathematical discourse, where different perceptions represent different objects of the category, provides tools of scrutiny for dealing with all perceptions. Structural similarities among perceptions may be studied, yet leaving ample room for differences and variety. This serves as a basis for a mathematical theory. It provides a way of extending and systematizing certain intuitive pre-theoretical conceptions about perception, about improving and/or completing an agent's perceptual grasp, about transition and comparison between various perceptions, etc. We shall also discuss a few example applications of the theory.

Mathematical formal systems are based on semantic primitives that are context independent. However, there is more than often an intuitive grounding for mathematical semantic primitives as well. This applies to this work where the formal system is grounded intuitively in a perceptual, cognitive environment.

All definitions, constructions and results are operated within the formal mathematical framework. This ensures a tidy treatment that introduces to the related domains tools of mathematical rigor and results that are meticulously stated. On the other hand, whenever a result is reached, it may be examined with regard to our grounding intuitions, pre-theoretical conceptions, and existing theories and opinions about artificial perception and related cognitive processes.

2 Background, Motivation, Contextualization

Newell states in [28] that perception is an area which should definitely be covered by theories of cognition, since cognitive behavior is a function of the environment. Nilsson proposes in [30] to work towards what he calls *Habile Systems*: programs of general, humanlike, competence. The abilities of such systems should ... *"include whatever is needed for an agent to get information about the environment in which it operates ... perceptual processing ... facilities for receiving, understanding, and generating communications ... ability to learn ..."*. This direction was already suggested in [29]. The proposed formalization of artificial perceptions and the tools it offers may constitute a step towards achieving the challenge of general basic artificial intelligence.

Application of mathematical methods for purposes of AI is not new [7, 12]. The advantages of mathematical formalizations as analyzed, for example, in the introduction to [11] include clarity, precision, versatility, generalizability, testability, allowance to model complex phenomena that are far too complex to be grasped by a verbal description, and allowance to use results of a well developed science.

Within mathematics, category theory seems suitable for purposes of AI. This science tries, in a sense, to approximate intelligence by creating particular models of artificial intelligence as well as foundations for a general account of such intelligence. In that context the following quotation from Lawvere [22] seems relevant: 'Even within mathematical experience, only that [category] theory has approximated a *particular* model of the general, sufficient as a foundation for a *general* account of all particulars'. Lawvere further argues that category theory provides a guide to the complex, but very non-arbitrary constructions of the concepts and their interactions which grow out of the study of *any* serious object of study.

There has not been, however, much AI-related research utilizing mathematical category theory. Banerfi [5] employs categorical terminology and tools for problem solving strategies. He defines a strategy to be a functor from the category of problems to the category of solutions. Two strategies are considered similar if there is a natural transformation of one into the other. Hodgson [17] remarks that category theory tends to bring out those aspects of a situation that are in some sense more natural than others, and follows a similar approach for purposes of problem solving with problem decomposition. Category theory is used to relate problems, features and solutions. In particular, given two sets of features on a problem, it is proposed to take their union, namely their pushout, with the idea that if one can reduce each set of features to their desired values, perhaps there is some way in which it can be done for the union. This is similar in some sense to our application of a pushout.

Lowry [23, 24] makes use of category theory emphasis on mappings between structures, and employs category theoretical tools for purposes of problem reformulation. He introduces a natural definition of isomorphic reformulation which links isomorphic transformations on models with isomorphic transformations on theories. His notion of an isomorphism between two categories of models simplifies the mathematics of problem reformulation and clears up confusions.

Zimmer [33] employs category theory as the mathematical reasoning for a study of representation engineering and of problem decomposition. His idea is to have one kind of algebraic object and one type of algebraic morphism that counts as a representation. He views domains as concrete categories and he shows how this approach provides useful tools for a representation engineering system.

A few more examples of the use of category theory in AI are described in a review by Benjamin [8]. The categories in these and other applications are different from the category of artificial perceptions. They share, however, the need for category theory in cases where it is necessary to generalize things to a point where only those aspects of a phenomenon that are (in some sense) natural and essential stand out.

In the context of perception and cognition categorization is typically applied at the level of an agent that categorizes its environment. A perceptive cognitive agent activates categorical constructions to analyze objects and relationships in its environment. In a certain sense this study employs a similar pattern at a higher level. Consider an environment that consists of such agents, abstracted as perceptions. AI researchers, being perceptive and cognitive agents, are invited to practise category theory for the circumscription and the study of what is universal in this environment. This is done in terms of very few primitives: object, morphism, domain and codomain of a morphism, and composition. Magnan and Reyes [26] analize category theory as a conceptual tool in the study of cognition, characterizing the properties that may be formulated by these primitives as the 'universal properties'.

Two other attempts at formalizing aspects of perception and cognition are by Gärdenfors [15] and Indurkhya [18]. They share some common aspects with the formalizations of this study, however they do not bring into play category theoretical constructions. It is precisely the categorical toolkit that enables one to model more complex cognitive constructions and processes by using the results of a well developed science. Morphisms, products and coproducts, pullbacks and pushouts are applied here as an example. In further applications we lay hands on more substantial tools: natural transformations and free functors. The versatility of the categorical tools provides a viable infrastructure.

Dealing with alternative viewpoints is still a knotty point for AI. Researchers in the area of ontology design agree that achieving interoperability and sharing of independently created ontologies is a challenging task [14]. A categorical setting where perceptions represent, for instance, different ontologies could provide tools for interoperability and sharing in the form of perception morphisms, products, coproducts, etc. Additional motivating applications are proposed in later sections.

3 Perceptions as Mathematical Objects

We postulate the abstract idea of a perception as a mathematical construct which relates between phenomena outside the artificial agent, a set of *outside world elements*, and reflections which are internal to the artificial agent, a set of *connotations*. Every perception has its own set of world elements, its own set of connotations, and its own predicative correspondence between the sets.

Definition 1 *A* Perception *is a 3-tuple* $\langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ *where:*

- $\mathcal{E}$ *and* $\mathcal{I}$ *are finite, disjoint sets.*
- ϱ *is a 3-valued predicate* $\varrho : \mathcal{E} \times \mathcal{I} \rightarrow \{\mathtt{t}, \mathtt{f}, \mathtt{u}\}$

The set $\mathcal{E}$ represents the outside, objective, world which is perceived. Anything which exists independent of the artificial agent, and could perhaps be discerned by it, could be a legitimate element of $\mathcal{E}$, and hence a *world element*. Example world elements may be a sound, a light, a blow of wind, a vapor (smelly or not), a candy bar, a glass of wine, etc. These example world elements are typically discerned by the human sensory-motor apparatus, but some artificial agents may be unable to discern them. These agents may, however, discern world elements that are imperceptible for humans, such as certain kinds of radiation. Furthermore, different perceptions might break the same reality into distinct, integral, world elements. As an example, wherever one agent perceives one world element ‘box’, another agent may perceive an arrangement of six world elements ‘board’. For humans, a human face would usually be a single world element that is easily perceived. Whether this is also the case where artificial perception is involved, is, however, not so clear. Hence, although we assume the external environment to have an objective existence, its division into world elements depends on the specific perception. (This phenomenon, as related to humans, has been studied by gestalt psychology [13]).

The set $\mathcal{I}$ stands for the internal representation of world elements. Its elements have a subjective existence dependent on the perception. Anything which may be stored and manipulated internally (words, symbols, icons, etc.) could be a legitimate element of $\mathcal{I}$, and hence a *connotation*. Example connotations may stand for the pitch and/or duration and/or timbre and/or volume of a sound, the brightness and/or hue and/or saturation of a light, etc. These example connotations typically represent attributes or properties that are measurable by humans, and hence considered ‘objective’. However, ‘hot’ and/or ‘dark’ and/or ‘good’ and/or ‘?!?!?’ are legitimate connotations as well (the last one is not a typo). These are definitely not ‘objective’, they depend on the specific perception.

The three-valued predicate ϱ is the *Perception Predicate* (*p-predicate* for short), which relates world elements and connotations, the connection between the outside world and internal representations. The terminology for the various p-predicate values will be the following: if $\varrho(\mathrm{w}, \alpha) = \mathtt{t}$ then w *has connotation* α, if $\varrho(\mathrm{w}, \alpha) = \mathtt{f}$ then w *lacks connotation* α, and if $\varrho(\mathrm{w}, \alpha) = \mathtt{u}$ then w *may either have or lack this connotation*. The last, undefined, value might eventually become defined but right now it is not.

Remark 1 *For now, it may seem sufficient to describe* ϱ *as a mere partial function into* $\{\mathtt{t}, \mathtt{f}\}$*. Further applications (see section 9) justify the three valued setting and the designation of* ϱ *as a ‘predicate’.*

Perception, and the values of the p-predicate in particular, is part of the definition of an agent, given data. This is supposed to capture the intuition that perception is innate to the agent: its gestalt perception, mental imagery, sensory-motor apparatus, function, internal organization etc. An analog approach to human perception is practised by Lakoff [21]. Lakoff argues that the structures used to put together our conceptual systems grow out of bodily experience, as well as physical and social experience, and make sense in terms of it. As an example: sensory perception of what is *dark* or *big* is individual to every human. It follows that the question whether or not the value of the p-predicate is 'correct' is meaningless in this case. Connotations that are alphabetic strings do not necessarily follow their dictionary definitions (if they have any). A smelly invisible vapor may, for instance, have the connotation 'pink'. This may depend on the agent's own individual architecture, programming and experience. Likewise, the issue of *why* the p-predicate has any one of the three values at a certain point simply warrants no discussion. As an example, the undefined, $\mathtt{u}$, value of perception may be due to ignorance, irrelevance, future contingence or other reasons. From the philosophical point of view, these possible reasons are quite different one from the other. In our context, however, the actual reason for a specific $\mathtt{u}$ value, or whether or not it is already 'decided' in some transcendental way, is irrelevant.

4 Example Perceptions

The variety of primary elements of a musical sound (pitch, duration, timbre, volume etc.) gave rise to a variety of musical notation systems. None of them can handle all these elements with precision. The notation system that is internalized by an agent, would affect the way it perceives music. The examples below are, however, extremely simple. They are just meant to illustrate the concepts of this study. The example set of world elements consists of the 88 keys of a perfectly tuned piano: $\mathcal{E} = \{\mathrm{w}_\imath\}_{i=1}^{88}$, ordered from left to right. A key can be connoted, among others, by its location, by the pitch of its sound, by its color, etc.

Example 1 *$\mathcal{P}_1 = \langle \mathcal{I}_1, \varrho_1 \rangle$ is inspired by modern alphabetical pitch solmization.*
$\mathcal{I}_1 =$
$\{A\_flat, A, A\_sharp, B\_flat, B, C, C\_sharp, D\_flat, D, D\_sharp, E\_flat, E, F, F\_sharp, G\_flat, G, G\_sharp\}$.
The p-predicate is the expected one: $\varrho_1(\mathrm{w}, \alpha) = \mathtt{t}$ if and only if w sounds α, and $\varrho_1(\mathrm{w}, \alpha) = \mathtt{f}$ if and only if w does not sound α. ϱ_1 is totally two valued.

Example 2 *$\mathcal{P}_{1a} = \langle \mathcal{I}_{1a}, \varrho_{1a} \rangle$ is a 'C-Major perception':*
$\mathcal{I}_{1a} = \{A, B, C, D, E, F, G\}$.
It is similar to $\mathcal{P}_1$, except that it recognizes the white keys only. For all connotations $\alpha \in \mathcal{I}_{1a}$, and for all white keys w $\varrho_{1a}(\mathrm{w}, \alpha) = \varrho_1(\mathrm{w}, \alpha)$. For all connotations $\alpha \in \mathcal{I}_{1a}$, and for all black keys w $\varrho_{1a}(\mathrm{w}, \alpha) = \mathtt{u}$ (this perception 'features agnosia of the black keys').

Example 3 *$\mathcal{P}_2 = \langle \mathcal{I}_2, \varrho_2 \rangle$ is inspired by European pitch solmization:*
$\mathcal{I}_2 = \{Do, Re, Mi, Fa, Sol, La, Si, Ti\}$.
The p-predicate ϱ_2 behaves essentially like ϱ_{1a}. For all keys $\mathrm{w} \in \mathcal{E}$:

$\varrho_2(\mathrm{w}, \mathit{Do}) = \varrho_{1a}(\mathrm{w}, C)$, $\varrho_2(\mathrm{w}, \mathit{Re}) = \varrho_{1a}(\mathrm{w}, D)$, $\varrho_2(\mathrm{w}, \mathit{Mi}) = \varrho_{1a}(\mathrm{w}, E)$,
$\varrho_2(\mathrm{w}, \mathit{Fa}) = \varrho_{1a}(\mathrm{w}, F)$, $\varrho_2(\mathrm{w}, \mathit{Sol}) = \varrho_{1a}(\mathrm{w}, G)$, $\varrho_2(\mathrm{w}, \mathit{La}) = \varrho_{1a}(\mathrm{w}, A)$,
$\varrho_2(\mathrm{w}, \mathit{Si}) = \varrho_2(\mathrm{w}, \mathit{Ti}) = \varrho_{1a}(\mathrm{w}, B)$.

Example 4 $\mathcal{P}_3 = \langle \mathcal{I}_3, \varrho_3 \rangle$ *is a visual black and white perception:* $\mathcal{I}_3 = \{\mathit{white}, \mathit{black}\}$. *The p-predicate* ϱ_3 *is the expected totally two valued perception:* $\varrho_3(\mathrm{w}, \mathit{white}) = \mathtt{t}$ *and* $\varrho_3(\mathrm{w}, \mathit{black}) = \mathtt{f}$ *if and only if* w *is a white key, and vice versa for black keys.*

Other perceptions of the given $\mathcal{E}$ can be conceived, including connotations representing frequencies in Hz units, serial locations (e.g. $\varrho(\mathrm{w}_9, 9) = \mathtt{t}$), connotations such as *out of tune*, etc.

5 Perception Morphisms

Morphisms are formal mathematical descriptions of paths between mathematical objects that have a similar structure. Perception morphisms will describe structure preserving transitions between perceptions. In the present context we consider perception morphisms between perceptions of *the same outside environment.*

Definition 2 *Let* $\mathcal{E}_0$ *be a set of world elements.* Perceptions of $\mathcal{E}_0$, *designated* $\mathcal{P}rc_{\mathcal{E}0}$, *is the set of all perceptions* $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ *such that their first component* $\mathcal{E} = \mathcal{E}_0$.

Since the outside world is fixed, we shall omit the first component from the designation of perceptions in $\mathcal{P}rc_{\mathcal{E}}$: $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ is an element of $\mathcal{P}rc_{\mathcal{E}}$, a short notation for $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$. In the rest of this paper the discussion assumes a given set of world elements $\mathcal{E}$. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$ be two perceptions in $\mathcal{P}rc_{\mathcal{E}}$. A perception morphism from $\mathcal{P}$ to $\mathcal{Q}$ will be defined as a set mapping of the connotations. However, this 'translation' between connotations should 'make sense': The essence of connotations as meaningful representations of the outside world should be maintained. One thus needs to define some 'structure preservation' condition on the mapping.

Definition 3 *Let* $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ *and* $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$ *be two perceptions in* $\mathcal{P}rc_{\mathcal{E}}$. $f : \mathcal{P} \to \mathcal{Q}$ *is a* Perception Morphism, p-morphism *for short, if both of the following conditions hold:*

1. f *is a set mapping of the connotations* $f : \mathcal{I} \to \mathcal{J}$, *and*
2. f *is* No-Blur*: for all* $\mathrm{w} \in \mathcal{E}$, *and for all* $\alpha \in \mathcal{I}$, $\varrho(\mathrm{w}, \alpha) \neq \mathtt{u}$ *implies that* $\tau(\mathrm{w}, f(\alpha)) = \varrho(\mathrm{w}, \alpha)$

Hence the 'preserved structure' consists of the definite $(\mathtt{t},\mathtt{f})$ values of the p-predicate. By the no-blur condition the mapping is grounded in the perception of the environment $\mathcal{E}$. This is the essence of the p-morphism as a structural element in a category. Example p-morphisms will be given in a short while.

6 The Category of Perceptions of $\mathcal{E}$

Having defined perceptions and perception morphisms, we would like to define the *Category of Perceptions* as a basis for a mathematical theory of artificial perceptions. In the same manner the infrastructure for group theory is provided by defining groups, group homomorphisms, and the category of groups. The definition of a category requires that:

- One is given a set of *objects*.
- Given any pair of objects $\mathcal{P}, \mathcal{Q}$, one has a collection of *morphisms* $f : \mathcal{P} \to \mathcal{Q}$ from $\mathcal{P}$ to $\mathcal{Q}$. Given a morphism such as f, $\mathcal{P}$ is the *domain* of f, and $\mathcal{Q}$ is the *codomain* of f.
- Morphisms should be closed under composition: Given two morphisms $f : \mathcal{P} \to \mathcal{Q}$ and $g : \mathcal{Q} \to \mathcal{R}$, where the codomain of f is the same as the domain of g, one may form their *composite*, $f \circ g$, which is a morphism: $f \circ g : \mathcal{P} \to \mathcal{R}$, such that $f \circ g(a) = g(f(a))$ (i.e. apply f, then g).
- Composition should be associative: $f \circ g \circ h = (f \circ g) \circ h = f \circ (g \circ h)$.
- For every object $\mathcal{P}$ there should be an identity morphism $Id_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$.
- The identity morphism should be the (left and right) unit element of composition: For every $f : \mathcal{P} \to \mathcal{Q}$, $Id_{\mathcal{P}} \circ f = f = f \circ Id_{\mathcal{Q}}$.

In our context the objects are perceptions $\mathcal{P}, \mathcal{Q} \ldots$ and morphisms are p-morphisms. The remaining requirements still need to be settled. Composition of p-morphisms is defined by set composition of the mappings:

Definition 4 *Let $f : \mathcal{P} \to \mathcal{Q}$ and $g : \mathcal{Q} \to \mathcal{R}$ be two p-morphisms. Their* composite *$f \circ g : \mathcal{P} \to \mathcal{R}$ is the mapping defined, for every $\alpha \in \mathcal{I}$, by $\alpha \mapsto g(f(\alpha))$ (i.e. apply f, then g).*

It is easy to see that the composite of p-morphisms is no-blur and hence a p-morphism. Also, composition is clearly associative. Lastly, the identity p-morphism is defined in the obvious way:

Definition 5 *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception in $\mathcal{P}rc_{\mathcal{E}}$. The* Identity p-morphism *on $\mathcal{P}$ is the morphism $Id_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ such that $\forall \alpha \in \mathcal{I}$ $Id_{\mathcal{P}}(\alpha) = \alpha$.*

The identity p-morphism is trivially no-blur. It is clearly the (left and right) unit element of composition.

Proposition 1 *$\mathcal{P}rc_{\mathcal{E}}$, together with definition 3 of p-morphisms, definition 4 of composition, and definition 5 of the identity p-morphism, is a category.*

There are various possible sets $\mathcal{E}$ of world elements, so one is actually discussing a family of categories. This categorical context provides a well known mathematical environment within which one can scrutinize artificial perception and related cognitive processes.

Remark 2 *For readers who are interested in algebraic context: Following remark 1, if perceptions were viewed as mere partial arrows $\varrho : \mathcal{E} \times \mathcal{I} \to \{\mathtt{t}, \mathtt{f}\}$, then the category of perceptions could be obtained from the category $\mathcal{K}_{\mathcal{E}}$ of pairs*

$(\mathcal{E},\mathcal{I})$ by a comma-like construction [1]*, where p-morphisms between two arrows $g : \varrho_1 \to \varrho_2$ would be those $\mathcal{K}_{\mathcal{E}}$-maps $g : (\mathcal{E},\mathcal{I}_1) \to (\mathcal{E},\mathcal{I}_2)$ for which $\varrho_1 = g \circ \varrho_2$, and this condition would then replace the no-blur condition on p-morphisms. There are other possible variants to this formulation, all featuring certain mathematical elegance. The definitions that are used are, however, suitable for a wider audience. Also, a total arrow with three boolean values is more convenient for some applications.*

Implementing some basic category-theoretical concepts and properties in $\mathcal{P}rc_{\mathcal{E}}$ sets the premises for a mathematical theory: Artificial Perception Theory.

7 Theory of Artificial Perceptions: Basics

7.1 P-morphisms Scrutinized

Morphisms are primitives of any category theoretical construction, such as the ones that are used for the example applications in later sections. In themselves, p-morphisms are capable of capturing a variety of inter-agent as well as intra-agent cognitive processes. Their most obvious application is as a tool of interpretive transition and of comparison between different perceptions. Various categorical properties of these morphisms are capable of meticulously pointing out the nature of the transition. By the definition of p-morphisms, the codomain perception may be 'better' then the domain perception in various ways.

1. A p-morphisms may feature *Expansion*: There may be connotations in the codomain that are not images of connotations in the domain. This happens if and only if the mapping is not onto. In that case the p-morphism maps the domain connotations into an 'expanded', richer perception.

2. A p-morphism may feature *Unblurring*. By the no-blur (structure preservation) condition on p-morphisms, a p-morphism may introduce definite (`t`,`f`) values in the p-predicate of the codomain instead of some undefined (`u`) values of the domain perception. Intuitively, it may somehow 'sharpen the picture'. Figuratively speaking, when asked whether a certain world element has a certain connotation, the codomain perception will give an 'I don't know' answer fewer times then the domain perception.

3. A p-morphism may feature *Generalization*. As a mapping of elements, a morphism may render some differences between connotations indistinguishable. This happens when the mapping is many-to-one.

These are, in a sense, the three 'primitives' that pave a path between perceptions. Expansion and generalization were just related to the two primitives of set mappings: *onto* and *one-to-one*. Quite a few properties of a category (such as $\mathcal{P}rc_{\mathcal{E}}$) where the morphisms are based on set mappings are typically expected. The element that provides 'color' to this category is the no-blur condition on perception morphisms, driven by the three-valued setting. It is the third 'primitive' of p-morphisms, and it is relevant to many basic categorical constructions in $\mathcal{P}rc_{\mathcal{E}}$.

[1] Comma (or slice) categories are discussed in [25, p.46],[16, p.29],[6, p.35].

7.2 Example P-morphisms

The following examples involve p-morphisms between the example perceptions of the keys of a piano: $\mathcal{P}_1$ of example 1 (modern alphabetical pitch solmization), $\mathcal{P}_{1a}$ of example 2 ('C-Major perception'), and $\mathcal{P}_2$ of example 3 (European pitch solmization).

Example 5 *$h : \mathcal{P}_2 \rightarrow \mathcal{P}_1$ is defined by: $h(Do) = C$, $h(Re) = D$, $h(Mi) = E$, $h(Fa) = F$, $h(Sol) = G$, $h(La) = A$, $h(Si) = h(Ti) = B$.*

- *h is a legitimate p-morphism since it preserves the defined ($\mathtt{t}$,$\mathtt{f}$) values of perception (by the definitions of ϱ_1 and ϱ_2).*
- *h features expansion since it is not onto: $\mathcal{P}_1$ also has connotations for flats and sharps.*
- *h features unblurring: for a black key* w *and for all $\alpha \in \mathcal{I}_2$ $\varrho_2(\mathrm{w}, \alpha) = \mathtt{u}$, but $\varrho_1(\mathrm{w}, h(\alpha)) = \mathtt{f}$.*
- *h features generalization since $h(Si) = h(Ti) = B$.*

Example 6 *$h : \mathcal{P}_2 \rightarrow \mathcal{P}_{1a}$ is defined by the same mapping as in example 5. However, there is no expansion, no unblurring, and just a generalization by mapping both* Si *and* Ti *to the same element B. Indeed, these two perceptions are quite similar, in spite of the different solmization.*

Example 7 *Let $\mathcal{P}_{1b}$ be similar to $\mathcal{P}_{1a}$ of example 2, except that for all black keys* w *$\varrho_{1b}(\mathrm{w}, \alpha) = \mathtt{f}$. (It perceives that none of the black keys has any of the pitch connotations A through G). Let $h : \mathcal{P}_{1a} \rightarrow \mathcal{P}_{1b}$ be defined by the identity mapping on $\mathcal{I}_{1a}$. h is both onto and one-to-one (neither expanding not generalizing), but it is unblurring: $\mathcal{P}_{1b}$ can definitely perceive more than $\mathcal{P}_{1a}$.*

Example 8 *Let $\mathcal{P}_{1b}$ be defined as in example 7 above. $h : \mathcal{P}_{1b} \rightarrow \mathcal{P}_1$ is defined, for all $\alpha \in \mathcal{I}_{1b}$, by $h(\alpha) = \alpha$. It features expansion (into a perception with connotations for flats and sharps), but neither unblurring nor generalization.*

Although the connection between these pitch perceptions and the visual, black and white, perception of example 4 is clear, it seems hard to express a p-morphism between them with the mathematical toolkit presented so far. We need to be able to state, for example, that a key has the connotation *black* whenever it has *any one* of the *flat/sharp* connotations. For this case, as well as many others, we need to analyze, in mathematical terms, how three valued predicates may be embedded in boolean algebras. This is a further application that will be mentioned later.

7.3 Rigid P-morphisms

First, let us see what happens if we remove the unblurring property from morphisms. There is a subset of p-morphisms that preserve the structure of perception in a *rigid* fashion:

Definition 6 *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. Let $f : \mathcal{I} \rightarrow \mathcal{J}$ be a mapping of the connotations. f is a* Rigid p-morphism *if, for all world elements $\mathrm{w} \in \mathcal{E}$, and for all connotations $\alpha \in \mathcal{I}$, $\varrho(\mathrm{w}, \alpha) = \tau(\mathrm{w}, f(\alpha))$.*

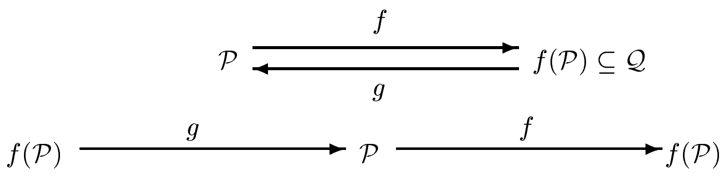

Figure 1: Characterization of 'Rigid' with arrows

In our terminology, rigid p-morphisms are not unblurring. They may be generalizing or expanding. The p-morphisms in examples 6 and 8 are rigid.

Recall that p-morphisms are formalizations of paths between perceptions. Unblurring essentially causes the path to be one-way, while rigidity means that one might be able to use the same path in both directions. The way of stating this categorically is by characterizing rigidity with left inverses that are also (rigid) p-morphisms. Let $f : \mathcal{P} \rightarrow \mathcal{Q}$ be a p-morphism, where $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. We designate by $f(\mathcal{I})$ the set of image connotations $f(\mathcal{I}) = \{\beta \in \mathcal{J} \| \exists \alpha \in \mathcal{I} \text{ such that } \beta = f(\alpha)\}$, and by $f(\mathcal{P})$ the image perception $f(\mathcal{P}) = \langle f(\mathcal{I}), \tau \|_{\mathcal{E} \times f(\mathcal{I})} \rangle$.

Proposition 2 *Let $f : \mathcal{P} \rightarrow \mathcal{Q}$ be a p-morphism. f is rigid if and only if for all choice functions $g : f(\mathcal{P}) \rightarrow \mathcal{P}$ such that $g \circ f = Id_{f(\mathcal{P})}$, g is a p-morphism. In that case g is rigid as well. (See figure 1).*

Proof. A. f is a p-morphism, hence whenever $\varrho(\mathrm{w}, \alpha)$ is defined, `t` or `f`, then so is $\tau(\mathrm{w}, f(\alpha))$, and they agree. B. If g is a p-morphism then whenever $\tau(\mathrm{w}, f(\alpha))$ is defined, `t` or `f`, then so is $\varrho(\mathrm{w}, \alpha)$, and they agree. It follows from A. and B. that $\tau(\mathrm{w}, f(\alpha))$ is defined, `t` or `f`, if and only if so is $\varrho(\mathrm{w}, \alpha)$, and they always agree. Therefore f is rigid. □
Rigidity will be a necessary property for most 'special' p-morphisms (e.g. p-isomorphisms, p-equalizers, p-coequalizers that are introduced below, and applied to cognitive processes later).

Left inverses are, indeed, not unique. A path between perceptions that can be traversed backwards by a unique p-morphism should obviously be one-to-one in addition to rigid, and it will be captured by the dual concept of *co-rigid*. Technically, this will later enable us to obtain results by the duality principle for categories. (Such dual results are typically obtained by using $\mathcal{K}^{\mathrm{op}}$, the mirror category of $\mathcal{K}$. $\mathcal{K}^{\mathrm{op}}$ has the same basic objects, but instead of having arrows $\rightarrow$ which designate morphisms, it has arrows $\leftarrow$ which indicate reversed presentations of morphisms. If $\mathcal{W}$ is a construct defined for $\mathcal{K}$, then the dual of $\mathcal{W}$, co-$\mathcal{W}$, is the construct defined for $\mathcal{K}$ by defining $\mathcal{W}$ in $\mathcal{K}^{\mathrm{op}}$ and reversing all the arrows. If $\mathcal{T}$ is a theorem true for $\mathcal{K}$, then the dual of $\mathcal{T}$, obtained by reversing all the arrows of $\mathcal{T}$, is true for $\mathcal{K}^{\mathrm{op}}$, and so, since $(\mathcal{K}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{K}$, it is true for$\mathcal{K}$.). To settle the meaning of *co-rigid*, we reverse all the arrows in figure 1.

Definition 7 *A p-morphism $f : \mathcal{P} \rightarrow \mathcal{Q}$ is* co-rigid *if its right inverse $g : f(\mathcal{Q}) \rightarrow \mathcal{P}$ is a p-morphism, and $f \circ g = Id_{\mathcal{P}}$.*

Co-rigid is thus a meaningful concept only with one-to-one p-morphisms, the only ones that could have right inverses. In that case, however, there exists a unique right inverse *which is also the only left inverse*, hence:

Corollary 1 *A p-morphism is co-rigid if and only if it is both rigid and one-to-one.*

The duality principle and co-rigidity will, in a short while, facilitate the treatment of p-equalizers.

7.4 P-Isomorphisms

A p-isomorphism is, not unexpectedly, just a renaming of the connotations. It is neither generalizing, nor unblurring or expanding. P-isomorphisms provide a formal way of saying that two perceptions are essentially the same.

Proposition 3 *A p-morphism is a p-isomorphism if and only if it is rigid, one-to-one, and onto.*

Proof. Let f be a p-morphism. f has an inverse mapping if and only if it is one to one and onto. In proposition 2 we have shown that this inverse mapping is a p-morphism if and only if both f and its inverse are rigid. □

Corollary 2 *In $\mathcal{P}rc_{\mathcal{E}}$ there may be p-morphisms that are both one-to-one and onto but they are not p-isomorphisms.*

Corollary 2 is typical to category theory, but not all categories have example morphisms of this kind.

Example 9 *Let $\mathcal{P} = \langle\{\alpha\}, \varrho \equiv \mathtt{u}\rangle$ (i.e. one connotation that is perceived* u *for all world elements), let $\overline{\mathcal{P}} = \langle\{\overline{\alpha}\}, \overline{\varrho} \equiv \mathtt{t}\rangle$ (i.e. one connotation that is perceived* t *for all world elements), and let $f : \mathcal{P} \to \overline{\mathcal{P}}$ be defined by $f(\alpha) = \overline{\alpha}$. f is clearly no-blur, one-to-one and onto but it is not a p-isomorphism.*

The p-morphism of example 7 was also of that kind. It can be easily seen that this phenomenon is due to the unblurring property of general p-morphisms. Perception improvements, to be introduced in a short while, constitute frequent examples of non-isomorphisms that are one-to-one and onto.

Example 10 *Let $\mathcal{P}_{2a}$ be similar to $\mathcal{P}_2$ of example 3, except that the connotation Ti is missing. In that case $\mathcal{P}_{2a}$ and $\mathcal{P}_{1a}$ are isomorphic, and the isomorphism h is defined similar to h of example 5.*

7.5 The Total P-Subcategory

The notion of rigid p-morphisms naturally leads to a subset of perceptions in $\mathcal{P}rc_{\mathcal{E}}$ where only these p-morphisms may apply. These are the perceptions with a classical, totally two-valued p-predicate. Stating that a perception is total means, essentially, that it cannot be further improved by way of unblurring: with the given set of connotations it provides the 'best perception'.

Definition 8 *$\mathcal{P}rc_{\mathcal{E}}^{Tot}$ is the subset of all* Total Perceptions *$\langle\mathcal{I}, \varrho\rangle$, with a total 2-valued p-predicate: $\varrho : \mathcal{E} \times \mathcal{I} \to \{\mathtt{t}, \mathtt{f}\}$*

Examples 1 and 4 are examples of total perceptions.

Proposition 4 *$\mathcal{P}rc_{\mathcal{E}}^{Tot}$ with rigid p-morphisms is a subcategory of $\mathcal{P}rc_{\mathcal{E}}$.*

It is easy to see that p-morphisms between total perceptions are always rigid, and hence all p-morphisms between total perceptions that are in $\mathcal{P}rc_{\mathcal{E}}$ are also in $\mathcal{P}rc_{\mathcal{E}}^{\text{Tot}}$. (In category theoretical terminology $\mathcal{P}rc_{\mathcal{E}}^{\text{Tot}}$ is a *full* subcategory). It should be noted, however, that there may also be rigid p-morphisms between perceptions that are not totally two-valued, such as in example 6.

7.6 The Universal Perception

The total subcategory has one member which may be loosely described as the 'ultimate perception'. If a total perception means that it cannot be further improved by way of unblurring, then what if it cannot be further improved in any other way as well? Category theory provides a way for saying that 'one cannot get any further (namely *better*) than that' in the form of *a terminal object.*

Definition 9 *The* Universal Perception of $\mathcal{E}$, *designated* $\mathcal{U}_{\mathcal{E}} = \langle 2^{\mathcal{E}}, \epsilon \rangle$, *is a perception where*

- *The set of connotations, $2^{\mathcal{E}}$, is the field of all subsets of $\mathcal{E}$.*
- *For all $\mathrm{w} \in \mathcal{E}$, and for all $A \subseteq \mathcal{E}$, $\epsilon(\mathrm{w}, A) = \mathtt{t}$ if and only if $\mathrm{w} \in A$, otherwise $\epsilon(\mathrm{w}, A) = \mathtt{f}$.*

The Universal Perception of $\mathcal{E}$ thus has a totally two valued p-predicate, and hence $\mathcal{U}_{\mathcal{E}} \in \mathcal{P}rc_{\mathcal{E}}^{\text{Tot}}$. For any subset of world elements it has a unique connotation which describes it accurately. The universal perception of the example set of the keys of a piano has 2^{88} connotations: one for every subset of the 88 keys.

Proposition 5 *The universal perception is a terminal object in $\mathcal{P}rc_{\mathcal{E}}^{Tot}$.*

Proof. Recall that an object $\mathcal{T}$ in a category $\mathcal{K}$ is *terminal* if for every object $\mathcal{A}$ of $\mathcal{K}$ there is a unique morphism $\mathcal{A} \to \mathcal{T}$.
It is first shown that if there exists a p-morphism $\mathcal{P} \to \mathcal{U}_{\mathcal{E}}$ in $\mathcal{P}rc_{\mathcal{E}}^{\text{Tot}}$ then it is unique. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and assume two p-morphisms $f_1, f_2 : \mathcal{P} \to \mathcal{U}_{\mathcal{E}}$. These p-morphisms are rigid: $\forall \mathrm{w} \in \mathcal{E}$, $\forall \alpha \in \mathcal{I}$, $\forall i = 1, 2$ $\epsilon(\mathrm{w}, f_\imath(\alpha)) = \varrho(\mathrm{w}, \alpha)$. This implies that $\forall \alpha \in \mathcal{I}$, $f_1(\alpha) = f_2(\alpha)$, and hence $f_1 = f_2$. To show that such a p-morphism always exists, define the *Natural p-morphism from a Total Perception into the Universal Perception*: $\eta : \mathcal{P} \to \mathcal{U}_{\mathcal{E}}$, by $\eta(\alpha) = \{\mathrm{w} \in \mathcal{E} \| \varrho(\mathrm{w}, \alpha) = \mathtt{t}\}$. It is easy to see that η is a rigid p-morphism. □

Corollary 3 *As a terminal object, the universal perception is unique up to isomorphism.*

Example 11 *The natural p-morphism from (the total) $\mathcal{P}_3$ of example 4 into the universal perception should map the connotation white to the subset of all (fifty two) white keys, and the connotation black to the subset of all (thirty six) black keys.*

Example 12 *The natural p-morphism from (the total) $\mathcal{P}_1$ of example 1 into the universal perception should map the connotation A to the subset of (eight) keys that sound A, the connotation A_flat to the subset of (seven) keys that sound A_flat, etc.*

The dual concept is an initial perception. Recall that an object $\mathcal{I}$ in a category $\mathcal{K}$ is *initial* if for every object $\mathcal{A}$ of $\mathcal{K}$ there is a unique morphism $\mathcal{I} \to \mathcal{A}$. The universal perception is not an initial object (and thus not a zero object, which is categorically defined as both initial and terminal). One half of the initial object property, the uniqueness of any p-morphism $f : \mathcal{U}_{\mathcal{E}} \to \mathcal{P}$, can be shown for $\mathcal{P}$'s in the subcategory $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Tot}}$ (the proof is essentially similar to the uniqueness proof of proposition 5), but not the existence. As a matter of fact, it can be easily seen that if such a p-morphism exists then it must be a p-isomorphism, defined similarly to η in the proof of proposition 5. Loosely speaking, the universal perception could not be generalized, unblurred or expanded any further - it is the 'best of the kind'.

As for the entire category $\mathcal{P}rc_{\mathcal{E}}$, one half of the terminal object property, the existence of a p-morphism from every perception into the universal perception, can be shown.

Proposition 6 *For every perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, there exists a p-morphism $f : \mathcal{P} \to \mathcal{U}_{\mathcal{E}}$.*

Proof. Define $\langle \mathcal{I}, \varrho' \rangle$ to be a total perception (i.e. in $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Tot}}$) such that $\varrho'(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \alpha)$ if and only if $\varrho(\mathrm{w}, \alpha) \neq \mathtt{u}$. This can be achieved from $\mathcal{P}$ by an arbitrary assignment of a definite truth value wherever $\varrho(\mathrm{w}, \alpha) = \mathtt{u}$. By the terminal property of $\mathcal{U}_{\mathcal{E}}$ in $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Tot}}$, there exists a unique morphism (the natural morphism) $\eta : \langle \mathcal{I}, \varrho' \rangle \to \mathcal{U}_{\mathcal{E}}$. Also, the p-morphism $f' : \mathcal{P} \to \langle \mathcal{I}, \varrho' \rangle$ defined by the identity mapping is no-blur by the definition of ϱ'. The desired p-morphism is thus $f = f' \circ \eta$. $\square$
However, f is not unique. Its definition as above involves a choice for every $\mathtt{u}$ value: should it become $\mathtt{t}$ or $\mathtt{f}$?, and the answer is, of course, not unique.

Example 13 *A p-morphism $f : \mathcal{P}_{1a} \to \mathcal{U}_{\mathcal{E}}$ as in proposition 6 above, could be expressed as the composite $f = h \circ \eta$, where h is the p-morphism of example 7, and η is defined similar to that of example 12.*

7.7 Improvements and Total Improvements

Having defined total perceptions and the universal perception as paradigms of 'good' perception, one might naturally ask how to get there. Improvements are arrows that formalize steps in that direction.

Definition 10 *An* Improvement p-morphism *is a p-morphism $f : \langle \mathcal{I}, \varrho \rangle \to \langle \mathcal{I}, \varrho' \rangle$ where f is the identity mapping on $\mathcal{I}$.*

An improvement p-morphism f is interesting whenever it is not rigid (otherwise one gets the identity p-morphism). Such an unblurring improvement could capture a case of 'learning': a proper improvement of perception within the same agent. The set of connotations is unchanged, only perception is unblurred. Such p-morphisms are both one-to-one and onto but they are not p-isomorphisms (because they are not rigid). Example 7 provides such a p-morphism.

In the case where ϱ' is a total p-predicate (i.e. the codomain of f is in the total subcategory $\mathcal{P}rc_{\mathcal{E}}^{\mathrm{Tot}}$, as in example 7), then we say that f is a *Total Improvement.* f' in the proof of proposition 6 is the typical total improvement. Given a perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, there is a one-to-one correspondence between all p-morphisms $f : \mathcal{P} \to \mathcal{U}_{\mathcal{E}}$ into the universal perception and all total improvements of $\mathcal{P}$. One side of the correspondence is given by proposition 6 above. To show how every $f : \mathcal{P} \to \mathcal{U}_{\mathcal{E}}$ implies a total improvement, define $f' : \langle \mathcal{I}, \varrho \rangle \to \langle \mathcal{I}, \varrho' \rangle$ using the identity mapping on $\mathcal{I}$, and $\varrho'(\mathrm{w}, \alpha) = \epsilon(\mathrm{w}, f(\alpha))$.

A total improvement of a given perception $\mathcal{P}$ (i.e. a p-morphism into the universal perception) may be regarded as 'a possible total perception' for that perception. It is similar in nature to the concept of 'possible worlds' as in [20]. Perception improvements and total improvements also provide a categorical formalization of the notion of the third truth value, u.

7.8 The Initial, Empty Perception

The categorical concept that is dual to a terminal object, namely to our 'ultimate' universal perception, is an initial object. Loosely, it stands for 'no perception'. Naturally, any perception is better than that, and hence there is always a path (i.e. a p-morphism) from 'no perception' to 'some perception'.

Proposition 7 *The initial object for $\mathcal{P}rc_{\mathcal{E}}$ is the* Empty Perception*: $\mathcal{P}_{\emptyset} = \langle \emptyset, \varrho_{\emptyset} \rangle$.*

Proof. For every perception $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle \in \mathcal{P}rc_{\mathcal{E}}$ there is the unique empty mapping of connotations $\emptyset \to \mathcal{I}$ which emptily stands the no-blur condition for p-morphisms. □

The variety of artificial perceptions of $\mathcal{E}$ thus lies between the initial perception with no connotations and the universal perception with $2^{|\mathcal{E}|}$ connotations. This includes all possible blurred perceptions, total perceptions, and variations of partially blurred perceptions in between. The number of different connotation will be considered in the context of *synonyms*.

7.9 P-Epimorphisms and P-Monomorphisms

The meaning of onto maps and one-to-one maps was already discussed in section 7.1 in the context of expansion and generalization. P-morphisms are defined in terms of set mappings of connotations, and in that case categorical epimorphisms are onto maps, and categorical monomorphisms are one-to-one maps. These facts will be settled now in order to conform with general category theoretical terminology. The following results are, naturally, similar to their equivalents in the category of sets. Minor adjustments of the proofs are needed to take care of the no-blur condition for p-morphisms.

Proposition 8 *A p-morphism is onto if and only if it is an epimorphism.*

Proof. Recall that a morphism f is an *epimorphism* if $f \circ g = f \circ h$ always implies that $g = h$. To show that onto implies epi the categorical set proof can be used (see, for example, [1, p.3]). To show that epi implies onto, let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and let $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$, and assume negatively that $f : \mathcal{P} \to \mathcal{Q}$ is not onto: There exists $\overline{\beta} \in \mathcal{J}$ such that $\forall \alpha \in \mathcal{I}$, $f(\alpha) \neq \overline{\beta}$. Let $\beta' \notin \langle \mathcal{J}, \tau \rangle$, and define a perception $\mathcal{R} = \langle \mathcal{J} \cup \{\beta'\}, \tau' \rangle$, where for all $\mathrm{w} \in \mathcal{E}$, $\tau'(\mathrm{w}, \beta') = \tau(\mathrm{w}, \overline{\beta})$ and, for all $\beta \in \mathcal{J}$, $\tau'(\mathrm{w}, \beta) = \tau(\mathrm{w}, \beta)$. We are now ready to define two p-morphisms $g, h : \mathcal{Q} \to \mathcal{R}$. g is the identity mapping on $\mathcal{J}$. $h(\beta') = \beta'$, and for all $\beta \in (\mathcal{J} - \{\beta'\})$, $h(\beta) = \beta$. It is easy to see that both g and h are no-blur (moreover: rigid), and thus p-morphisms. Also $g \neq h$, but $f \circ g = f \circ h$, in contradiction to the assumption that f is a p-epimorphism. □

Proposition 9 *A p-morphism is one-to-one if and only if it is mono.*

Proof. Recall that a morphism f is a *monomorphism* if $g \circ f = h \circ f$ always implies $g = h$. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and let $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. To show that one-to-one implies mono the categorical set proof can be used. To show that mono implies one-to-one, let $\mathcal{R} = \langle \{\alpha\}, \varrho \equiv \mathtt{u} \rangle$ (i.e. one connotation that is perceived u for all world elements), Assume negatively that f is not one-to-one: there exist $\alpha_1, \alpha_2 \in \mathcal{I}$ such that $\alpha_1 \neq \alpha_2$ yet $f(\alpha_1) = f(\alpha_2)$. Define $g, h : \mathcal{R} \to \mathcal{P}$ by $g(\alpha) = \alpha_1$ and $h(\alpha) = \alpha_2$. g, h are, of course, no-blur and hence p-morphisms. By the definitions $f(g(\alpha)) = f(h(\alpha))$, so that $g \circ f = h \circ f$, yet $g \neq h$, in contradiction to the assumption that f is a p-monomorphism. □
The p-morphism of example 6 is a p-epimorphism, and the p-morphism of example 8 is a p-monomorphism. These categorical concepts will be applied later (e.g. in the context of p-image factorization, p-products and p-coproducts).

7.10 P-Coequalizers and P-Equalizers

P-coequalizers and p-equalizers are special cases of p-morphisms that will be applied in the sequel (they are relevant to all the sections below).

P-coequalizers are, essentially, arrows that generalize similar connotations into one connotation and hence a practical tool in many situations.

Proposition 10 *A p-morphism is a coequalizer if and only if it is both rigid and epi.*

Proof. Recall that a morphism $f : \mathcal{P} \to \mathcal{Q}$ is a *Coequalizer* if there exists a pair $p_1, p_2 : \mathcal{R} \to \mathcal{P}$ such that $p_1 \circ f = p_2 \circ f$, and such that whenever $f' : \mathcal{P} \to \mathcal{Q}'$ satisfies $p_1 \circ f' = p_2 \circ f'$, then there is a unique morphism $\psi : \mathcal{Q} \to \mathcal{Q}'$ such that $f \circ \psi = f'$. In this situation f is called the *coequalizer of* p_1 *and* p_2. Categorically, every coequalizer is epi. We show that, in the category of perceptions, a p-coequalizer must be rigid. Assume negatively that f is a p-coequalizer, yet it is not rigid. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. Hence there exists a connotation $\alpha' \in \mathcal{I}$ and a world element $\mathrm{w}' \in \mathcal{E}$ such that $\varrho(\mathrm{w}', \alpha') = \mathtt{u}$ but $\tau(\mathrm{w}', f(\alpha')) \neq \mathtt{u}$. Define $\mathcal{Q}'$ and f' in the following way: $\mathcal{Q}' = \langle \mathcal{J}, \tau' \rangle$ (i.e. the same set of connotations as for $\mathcal{Q}$), and $\forall \alpha \in \mathcal{I}$, $f'(\alpha) = f(\alpha)$ but τ' is such that f' is rigid: $\forall \mathrm{w} \in \mathcal{E}$, $\forall \alpha \in \mathcal{I}$, $\tau'(\mathrm{w}, f'(\alpha)) = \varrho(\mathrm{w}, \alpha)$. By this definition, and since f is a p-coequalizer, $p_1 \circ f' = p_2 \circ f'$. Since f is assumed to be a p-coequalizer, then there exists a unique $\psi : \mathcal{Q} \to \mathcal{Q}'$. In particular, $\psi(f(\alpha')) = f'(\alpha')$, but $\varrho(\mathrm{w}, \alpha') = \mathtt{u}$ and $\tau(\mathrm{w}, f(\alpha')) \neq \mathtt{u}$. By definition, f' is rigid and hence $\tau'(\mathrm{w}, f'(\alpha')) = \tau'(\mathrm{w}, \psi(f(\alpha')) = \mathtt{u}$. So ψ is not a legitimate p-morphism by definition 3 (i.e. it is blurring), and one gets a contradiction. This completes the proof that every p-coequalizer is rigid as well as epi. (Loosely speaking, in order to be the *most general* p-morphism with the coequalizer property, a p-morphism has to be rigid.)

We show now that every rigid p-epimorphism is a p-coequalizer of any pair of its left inverses. Let f be a rigid p-epimorphism. Since f is onto, there exists a *choice function* $g : \mathcal{Q} \to \mathcal{P}$ such that $g \circ f = Id_{\mathcal{Q}}$. Since f is rigid every such choice function is a (rigid) p-morphism. Using the notation of proposition

2, let $\mathcal{R} = \mathcal{Q}$ and let G be the set of all these choice functions. Hence for all $g, h \in G$, $g \circ f = h \circ f = Id_{\mathcal{Q}}$, and $f \circ g \circ f = f \circ h \circ f = f$. If G happens to be a singleton, then f has a unique inverse and is thus an isomorphism, which trivially stands the coequalizer condition. If G has more than one element, let $g, h \in G$ be any two *distinct* p-morphisms in G. Assume that $f' : \mathcal{P} \to \mathcal{P}'$ is such that $g \circ f' = h \circ f'$. It follows that $f \circ g \circ f' = f \circ h \circ f' = f'$. Define $\psi = g \circ f' (= h \circ f')$, hence $f \circ \psi = f \circ g \circ f' (= f \circ h \circ f') = f'$. ψ is clearly the unique p-morphism with this property, since $\psi = g \circ f'$. This completes the proof that any rigid p-epimorphism f is a p-coequalizer. □
The p-morphism of example 6 is a p-coequalizer.

A p-equalizer is essentially an arrow that designates a subset of connotations, and hence a 'sub-perception'. Categorically, it is the concept dual to coequalizer. A morphism $f : \mathcal{P} \to \mathcal{Q}$ is an *Equalizer* if there exists a pair $p_1, p_2 : \mathcal{Q} \to \mathcal{R}$ such that $f \circ p_1 = f \circ p_2$, and such that whenever $f' : \mathcal{P}' \to \mathcal{Q}$ satisfies $f' \circ p_1 = f' \circ p_2$ there is a unique morphism $\phi : \mathcal{P}' \to \mathcal{P}$ such that $\phi \circ f = f'$. In this situation f is called the *equalizer of* p_1 *and* p_2. Categorically, every equalizer is mono.

Proposition 11 *A p-morphism is a p-equalizer if and only if it is a rigid p-monomorphism.*

Proof. Proposition 10 with reversed arrows yields a co-rigid p-monomorphism (see definition 7). By corollary 1 and proposition 9 it is equivalent to a rigid p-monomorphism. The proof follows by the duality principle for category theory (as explained in section 7.3). □
The p-morphism of example 8 is a p-equalizer.

Proposition 12 *Let* $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ *and* $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. *Two p-morphisms* $f_1, f_2 : \mathcal{P} \to \mathcal{Q}$ *have a p-coequalizer if and only if* $\forall \mathrm{w} \in \mathcal{E}$, $\forall \alpha \in \mathcal{I}$, $\tau(\mathrm{w}, f_1(\alpha)) = \tau(\mathrm{w}, f_2(\alpha))$.

Proof. In the category of sets a coequalizer is defined (for every two maps) by using *an equivalence relation* in the following way: Given $f_1, f_2 : \mathcal{P} \to \mathcal{Q}$, define on $\mathcal{J}$ the relation $R = \{(f_1(\alpha), (f_2(\alpha)) \| \alpha \in iI\}$. R' is defined as the smallest equivalence relation containing R: $(a, b) \in R'$ if and only if either $a = b$ or (a, b) can be linked by a chain $(a_1, ..., a_{n+1})$ of elements of $\mathcal{J}$ where $a_1 = a$ and $a_{n+1} = b$, and for every $1 \leq k \leq n$, either $(a_k, a_{k+1}) \in R$, or $(a_{k+1}, a_k) \in R$. The map $h : \mathcal{J} \to \mathcal{J}/_{R'}$, defined by $h(\alpha) = [\alpha]_{R'}$, is the coequalizer of f_1, f_2. To adapt this construction to the category of perceptions, the set of connotations $\mathcal{J}/_{R'}$ should come with a well defined p-predicate, say τ', and the mappings of connotations should yield no-blur p-morphisms. Assume first that $f_1, f_2 : \mathcal{P} \to \mathcal{Q}$ have a p-coequalizer. It was shown that the p-coequalizer h must be rigid. Also τ' should be well defined. It follows that, for all $\mathrm{w} \in \mathcal{E}$, and for all $\alpha \in \mathcal{I}$, $\tau(\mathrm{w}, f_1(\alpha)) = \tau'(\mathrm{w}, f_1 \circ h(\alpha)) = \tau'(\mathrm{w}, f_2 \circ h(\alpha)) = \tau(\mathrm{w}, f_2(\alpha))$. On the other hand assume that, for all $\mathrm{w} \in \mathcal{E}$, and for all $\alpha \in \mathcal{I}$, $\tau(\mathrm{w}, f_1(\alpha)) = \tau(\mathrm{w}, f_2(\alpha))$. In that case h, as defined above (using the general construction for the category of sets), is the desired p-coequalizer: it is easy to see that it is a rigid p-epimorphism. The proof continues along the same lines as for the category of sets (see, for example, [1, p.19,20]). □

Example 14 *Two rigid p-morphisms* $f_1, f_2 : \mathcal{P} \to \mathcal{Q}$ *always have a p-coequalizer.*

Proposition 13 *Two p-morphisms* $f_1, f_2 : \mathcal{P} \to \mathcal{Q}$ *always have a p-equalizer*

Proof. In this case the construction in the category of sets can be directly applied without change: Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$, and define $\mathcal{I}' = \{\alpha \in \mathcal{I} | f_1(\alpha) = f_2(\alpha)\}$, $\mathcal{R} = \langle \mathcal{I}', \varrho|_{\mathcal{E} \times \mathcal{I}'} \rangle$. The p-equalizer h of $f_1, f_2 : \mathcal{P} \to \mathcal{Q}$ is defined by the inclusion map $\mathcal{I}' \subseteq \mathcal{I}$, $h : \mathcal{R} \to \mathcal{P}$, which is, of-course, a rigid p-monomorphism. The proof continues along the same lines as in the category of sets (see, for example, [1, p.21,22]). □

Example 15 *If $\mathcal{I}' = \emptyset$ (i.e. there is no $\alpha \in \mathcal{I}$* such that *$f_1(\alpha) = f_2(\alpha)$), one gets a p-equalizer which is the empty inclusion of the initial, empty perception $\mathcal{P}_\emptyset = \langle \emptyset, \varrho_\emptyset \rangle$ of proposition 7.*

7.11 Analyzing the Path between Perceptions: Image Factorization

The next standard categorical construction provides, in our context, insight into the construction and the analysis of a path between perceptions. In section 7.1 expansion, unblurring, and generalization were indicated as the primitives constituting a p-morphism. Basic perceptual transitions that are captured in $\mathcal{P}rc_{\mathcal{E}}$ are described using these terms [2]. All possible eight combinations of the three primitives were already encountered within our categorical context:

- A general p-morphism may generalize, expand, unblur.
- A rigid p-morphism may not unblur.
- A p-epimorphism may not expand.
- A p-monomorphism may not generalize.
- An improvement p-morphism may only unblur.
- A p-coequalizer may only generalize.
- A p-equalizer may only expand.
- A p-isomorphism may not generalize, expand, or unblur.

There is a categorical way to fathom possible interactions between these three primitives in terms of the order in which they are applied. An *Image Factorization System* for a category consists of a pair (E, M), where E and M are classes of morphisms of the category satisfying the following four conditions:

1. Each one of E and M is closed under composition.
2. The morphisms in E are epi, and the morphisms in M are mono.
3. Every isomorphism is both in E and in M.
4. Every morphism $f : \mathcal{P} \to \mathcal{Q}$ has an $E - M$ factorization which is unique up to isomorphism. Namely, there exist $e_f \in E$ and $m_f \in M$ such that $f = e_f \circ m_f$. This factorization is unique in the sense that if $f = e \circ m$ is another such factorization, then there exists an isomorphism ψ (as shown in the commutative diagram of figure 2) with $e_f \circ \psi = e$ and $\psi \circ m = m_f$.

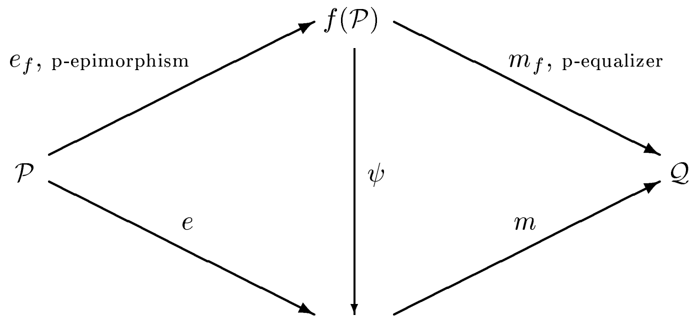

Figure 2: P-Image factorization

Proposition 14 *(p-epimorphisms,p-equalizers) is an image factorization system for $\mathcal{P}rc_{\mathcal{E}}$.*

Proof. The first three of the properties that are listed above obviously hold for E = p-epimorphisms and M = p-equalizers. We proceed to prove the fourth property. Let $f : \mathcal{P} \to \mathcal{Q}$ be a p-morphism such that $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. (The designations for the image set of connotations, $f(\mathcal{I})$ and the image perception $f(\mathcal{P})$ are the same as in the proof of proposition 2.) Define $e_f : \mathcal{P} \to f(\mathcal{P})$ by $e_f(\alpha) = f(\alpha)$ and $m_f : f(\mathcal{P}) \to \mathcal{Q}$ by $m_f(\beta) = \beta$. Obviously, $f = e_f \circ m_f$, e_f is onto (epi), and m_f is one-to-one (mono) and rigid, and hence, by proposition 11, a p-equalizer. Define: $E = \{e \| \exists f \text{ such that } e = e_f\}$ and $M = \{m \| \exists f \text{ such that } m = m_f\}$. To show that this factorization is unique up to isomorphism, consider another factorization $f = e \circ m$ with a p-epimorphism e and a p-equalizer m, such that $e : \mathcal{P} \to \mathcal{R}$ and $m : \mathcal{R} \to \mathcal{Q}$. Define: $\psi : f(\mathcal{P}) \to \mathcal{R}$ by $\forall \alpha \in \mathcal{I}$, $e_f(\alpha) \mapsto e(\alpha)$. Clearly, $e_f \circ \psi = e$ and $\psi \circ m = m_f$. Since both e_f and e are epi, so is ψ. Since both m_f and m are mono and rigid so is ψ. Hence ψ is a p-isomorphism. □

The p-image factorization system provides additional insight into the mechanism of p-morphisms. Generalization and unblurring are performed *together* by the first factor e, while expansion is performed separately by the second factor m.

Example 16 *An image factorization of $h : \mathcal{P}_2 \to \mathcal{P}_1$ of example 5 consists of:*

- *$e : \mathcal{P}_2 \to \mathcal{P}_{1b}$, which is a composition of the generalization of example 6 and the unblurring of example 7.*
- *$m : \mathcal{P}_{1b} \to \mathcal{P}_1$ as in example 8.*

The 'dual' system (coequalizers, monomorphisms), would have first performed the generalization, then would have followed with unblurring joined to the expansion. In the general case, however, unblurring should never be delayed until after the generalization, because the merging of two connotations forces the immediate unblurring of one of them whenever one is defined where the other is not.

Example 17 *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ be a perception, and let $\mathcal{U}_{\mathcal{E}}$ be the universal perception. The existence of $f : \mathcal{P} \to \mathcal{U}_{\mathcal{E}}$ was shown in proposition 6, using a composition $f = f' \circ \eta$. Unblurring is performed by f', while generalization and*

[2]More complex transitions are, in turn, described by categorical constructions that employ p-morphisms as *their* primitives.

expansion are performed by η. $f' \circ \eta$ is thus not *a category theoretical image factorization. It works because unblurring precedes generalization.*

8 Theory of Artificial Perceptions: Example Applications

8.1 Joining Perceptions Using Products and Coproducts

An intelligent agent should be able, among other things, to preserve its autonomy within a society of agents. (This bimodal aspect of cognitive behavior is discussed also by Newell [28].) The category of perceptions provides, among other things, convenient tools of scrutiny to formalize several forms of joint perception, with varying degrees of trust and partnership. We start with the least obliging one.

A coproduct of a family of perceptions is their 'minimal change common expansion' - an expansion of perception to include the perceptions of the other participants as well, with otherwise no unblurring or generalization.

Proposition 15 *The coproduct of a family $\{\mathcal{P}_\imath\}_{\imath=1}^n$ of perceptions is the perception*

$$\coprod_{\imath=1}^n \mathcal{P}_\imath = \langle \{(\alpha, i) \| \exists \imath \ (1 \le \imath \le n) \text{ such that } \alpha \in \mathcal{I}_\imath\}, \coprod_{\imath=1}^n \varrho_\imath \rangle$$

Where the set of connotations is the set coproduct (direct sum) of the sets of connotations, and the p-predicate $\coprod_{\imath=1}^n \varrho_\imath$ is defined by:
$\coprod_{\imath=1}^n \varrho_\imath(\mathrm{w}, (\alpha, \imath)) = \varrho_\imath(\mathrm{w}, \alpha)$*, with n injections $\nu_\imath : \mathcal{P}_\imath \to \coprod_{\imath=1}^n \mathcal{P}_\imath$, where $\nu_\imath(\alpha) = (\alpha, \imath)$.*

Proof. It is first observed that the injections are no-blur by the definition of the p-predicate $\coprod_{\imath=1}^n \varrho_\imath$, and are thus legitimate p-morphisms. By the category theoretical definition of coproducts it has to be shown that the change is minimal (note that the injections are p-equalizers: rigid and mono), namely that given any other perception $\mathcal{Q}$ similarly equipped with an indexed family of p-morphisms $\{f_\imath : \mathcal{P}_\imath \to \mathcal{Q}\}_{\imath=1}^n$, there exists a unique morphism $f : \coprod_{\imath=1}^n \mathcal{P}_\imath \to \mathcal{Q}$ such that $\forall \imath \ (1 \le \imath \le n) \ \nu_\imath \circ f = f_\imath$. Define f by $(\alpha, \imath) \mapsto f_\imath(\alpha)$. f is a p-morphism since the $f_\imath$'s are p-morphisms, and by the definition of the p-predicate $\coprod_{\imath=1}^n \varrho_\imath$. By its definition it is the only p-morphism such that $\forall \imath \ (1 \le \imath \le n) \ \nu_\imath \circ f = f_\imath$. $\square$

Corollary 4 *Since the set direct sum of the $\mathcal{I}_\imath$'s always exists, so does the coproduct perception.*

Every participating perception uses its own perception 'in the name of' the entire set of participating perceptions. If the coproduct has perception of some world element and connotation pair $(\mathrm{w}, (\alpha, \imath))$, then it bears meaning to the $\imath$'th participating perception. This kind of 'maximal trust' joint perception could be useful in any one of the many cases where there is more than one perception of a given environment. One possible example is spatial perception as analyzed in [10]. A system could use everyday, naive perception of points, places, bodies, holes etc. It could also use more sophisticated perceptions applying, for example, mathematical topology. In some cases it may be desirable to use all of them simultaneously, activating convenient connotations for different purposes, switching freely from one system to the other. (This is, perhaps, what humans do). In this case a coproduct perception as above may provide the formalization.

Example 18 *A coproduct of our example perceptions of the keys of a piano (examples 1, 2, 3, 4) would provide a perception that features internalization of modern alphabetical pitch solmization (with flats and sharps), as well as European pitch solmization, and also black and white visual perception. This does not mean, however, that the similarities and connections between these various perceptions are perceived or internalized. There is nothing in the joint coproduct perception that tells us about that right now. This issue will be discussed in a short while.*

A product of a family of perceptions is their 'minimal change common blur'. Perceiving is done by all the participants together, and all should agree on anything definite. Every participating perception is blurred exactly to the point where there is no conflict.

Proposition 16 *The product of a family $\{\mathcal{P}_\imath\}_{\imath=1}^n$ of perceptions is the perception*

$$\prod_{\imath=1}^n \mathcal{P}_\imath = \langle \prod_{\imath=1}^n \mathcal{I}_\imath, \prod_{\imath=1}^n \varrho_\imath \rangle$$

Where the set of connotations is the set product of the sets of connotations, and the p-predicate $\prod_{\imath=1}^n \varrho_\imath$ is defined as follows:

$$\prod_{\imath=1}^n \varrho_\imath(\mathrm{w}, (\alpha_1, ..., \alpha_n)) = \begin{array}{ll} \mathtt{t} & \text{if and only if } \forall \imath \ (1 \leq \imath \leq n) \ \varrho_\imath(\mathrm{w}, \alpha_\imath) = \mathtt{t} \\ \mathtt{f} & \text{if and only if } \forall \imath \ (1 \leq \imath \leq n) \ \varrho_\imath(\mathrm{w}, \alpha_\imath) = \mathtt{f} \\ \mathtt{u} & \text{otherwise} \end{array}$$

with n projections $\pi_\imath : \prod_{\imath=1}^n \mathcal{P}_\imath \to \mathcal{P}_\imath$, where $\pi_\imath(\alpha_1, ..., \alpha_n) = \alpha_\imath$.

Proof. It is first observed that the projections are no-blur by the definition of the p-predicate $\prod_{\imath=1}^n \varrho_\imath$, and are thus legitimate p-morphisms. By the category theoretical definition of products it has to be shown that the change is minimal (note that the projections are epi, with generalization and unblurring that are restricted to the necessary minimum), namely that given any other perception $\mathcal{Q}$ similarly equipped with an indexed family of p-morphisms $\{f_\imath : \mathcal{Q} \to \mathcal{P}_\imath\}_{\imath=1}^n$, there exists a unique morphism $f : \mathcal{Q} \to \prod_{\imath=1}^n \mathcal{P}_\imath$ such that $\forall \imath \ (1 \leq \imath \leq n) \ f \circ \pi_\imath = f_\imath$. Define f by $\alpha \mapsto (f_\imath(\alpha), ..., f_n(\alpha))$. f is a p-morphism since the $f_\imath$'s are p-morphisms, and by the definition of the p-predicate $\prod_{\imath=1}^n \varrho_\imath$. By its definition it is the only p-morphism such that $\forall \imath \ (1 \leq \imath \leq n) \ f \circ \pi_\imath = f_\imath$. □

Corollary 5 *Since the set product of the $\mathcal{I}_\imath$'s always exists, so does the product perception.*

Example 19 *A product of our example perceptions of the keys of a piano ($\mathcal{P}_1$ of example 1, $\mathcal{P}_{1a}$ of example 2, $\mathcal{P}_2$ of example 3, and $\mathcal{P}_3$ of example 4) would have, for instance:*

1. $\prod \varrho_\imath(\mathrm{w}_1, (A, A, La, white)) = \mathtt{t}$, *because:*
   $\varrho_1(\mathrm{w}_1, A) = \varrho_{1a}(\mathrm{w}_1, A) = \varrho_2(\mathrm{w}_1, La) = \varrho_3(\mathrm{w}_1, white) = \mathtt{t}$.
   (w_1 is the leftmost key of the piano, it is white and sounds A.)

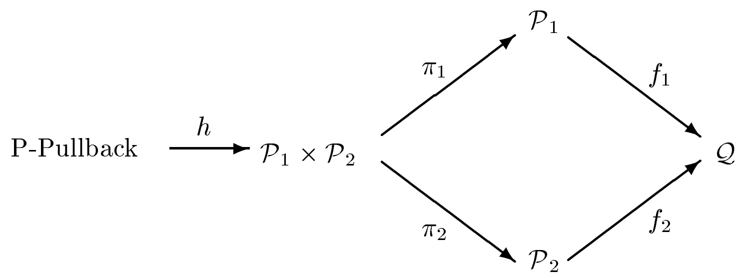

Figure 3: Minimal Trust Joint Perception with Compromise: P-Pullback

2. $\prod \varrho_\imath(\mathrm{w}_3, (A, D, Do, black)) = \mathtt{f}$, *because:*
$\varrho_1(\mathrm{w}_3, A) = \varrho_{1a}(\mathrm{w}_3, D) = \varrho_2(\mathrm{w}_3, Do) = \varrho_3(\mathrm{w}_3, black) = \mathtt{f}$.
(w_3 is the third key from the left, it sounds neither A, nor D, nor Do, and it is not black.)

3. $\prod \varrho_\imath(\mathrm{w}_2, (B\_flat, G, Fa, white)) = \mathtt{u}$, *because:*
$\varrho_1(\mathrm{w}_2, B\_flat) = \mathtt{t}$ *but* $\varrho_3(\mathrm{w}_2, white) = \mathtt{f}$.
(w_2 is the second key from the left, it does sound B_flat, but it is not white.)

4. $\prod \varrho_\imath(\mathrm{w}_2, (F, F, Fa, black)) = \mathtt{u}$, *because:*
w_2 *is a black key and hence* $\varrho_{1a}(\mathrm{w}_2, F) = \varrho_2(\mathrm{w}_2, Fa) = \mathtt{u}$, *(although there is an agreement that* $\varrho_1(\mathrm{w}_2, F) = \varrho_3(\mathrm{w}_2, white) = \mathtt{f}$.*)*

A product of perceptions is a 'minimum trust' joint perception, where all participants have to agree on every value of the p-predicate or else it is left undefined (in some cases all might have to be left undefined). Example cases where such a product perception may be useful are cases of 'diplomatic' negotiations between agents (as in [19]), where points of disagreement have to be blurred.

In the general case, most of the p-predicate values in a product perception are going to be undefined, since most connotation tuples would consist of essentially different connotations, as in the third instance of example 19 above. P-Pullbacks are capable of neatly restricting the product perception to the desired subset of connotations that feature some kind of agreement, obtaining a lax analog to set intersection. Categorically, a pullback is constructed out of a product and an equalizer. The intuitive idea behind p-pullbacks is to enhance partnership by selection (via the equalizer h in the commutative diagram of figure 3) of all the tuples where $\forall \imath\ \varrho_\imath(\mathrm{w}, \alpha_\imath) = \mathtt{t}$ (e.g. the first instance of example 19), or $\forall \imath\ \varrho_\imath(\mathrm{w}, \alpha_\imath) = \mathtt{f}$ (e.g. the second instance of example 19). Such tuples feature definite agreement. One may also include tuples where $\forall \imath\ \varrho_\imath(\mathrm{w}, \alpha_\imath) \neq \mathtt{t}$ (e.g. the fourth instance of example 19) or $\forall \imath\ \varrho_\imath(\mathrm{w}, \alpha_\imath) \neq \mathtt{f}$. Such tuples feature possible future agreement. To apply the theory to a specific case of 'minimal trust' joint perception: for any tuple of connotations that features a desirable (firm or future contingent) similarity, define the $f_\imath$'s of the diagram to point at the same element of (a trivially chosen) $\mathcal{Q}$, and consider the p-pullback of f_1, f_2, defined by the p-equalizer of $(h \circ \pi_1, h \circ \pi_2)$. It yields the subset of tuples from the product $\mathcal{P}_1 \times \mathcal{P}_2$, that feature similarities between the participants. Note that a p-equalizer could yield an empty inclusion (as in example 15). This could, for example, reveal the true poor nature of some 'minimal trust' joint perception: there is nothing meaningful to join.

This 'minimal trust joint perception with enhanced partnership' may be useful in the formalization of training situations as well. The (totally two-valued) perception of the trainer may be 'pulled back' with the perception of the trainee for a 'test'. Tuples that feature full agreement represent successful training. Tuples that feature possible agreement represent points where some more training is needed. Tuples that cannot be acomodated in the p-pullback represent failed training.

8.2 Synonyms and *-Perceptions

We now apply the category theoretical concepts of $\mathcal{P}rc_{\mathcal{E}}$ that were defined in this study for the formal casting of synonymity of connotations and related issues. Two connotations may be indistinguishable in that they stand for the same perception values.

Definition 11 *In a given perception* $\langle \mathcal{I}, \varrho \rangle$*, let* $\alpha, \beta \in \mathcal{I}$*.* α *and* β *are* Synonyms *(or* ϱ-synonyms*), denoted* $\alpha \simeq \beta$*, if, for all world elements* $\mathrm{w} \in \mathcal{E}$*,* $\varrho(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \beta)$*.*

Example 20 *In* $\mathcal{P}_2$ *of example 3 the connotations Si and Ti are synonyms.*

Example 21 *Since the piano is a well-tempered instrument, one gets in* $\mathcal{P}_1$ *of example 1:* $(C\_sharp) \simeq (D\_flat)$, $(D\_sharp) \simeq (E\_flat)$, $(F\_sharp) \simeq (G\_flat)$, $(G\_sharp) \simeq (A\_flat)$, $(A\_sharp) \simeq (B\_flat)$.

Proposition 17 $\simeq$ *is an equivalence relation (by the equality in the definition).*

Definition 12 *Let* $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$*. The* fully merged *or* reduced *connotation set for* $\mathcal{P}$ *is the quotient set* $\mathcal{I}^* = \mathcal{I}/_{\simeq}$ *.* $\mathcal{P}^* = \langle \mathcal{I}^*, \varrho \rangle$ *is a designation for* $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ *with a reduced connotation set. (*ϱ *is, of course, well defined on* $\mathcal{E} \times \mathcal{I}^*$ *as well.)* *-perceptions *are perceptions with a reduced connotation set.*

The passage from a perception to its reduced version formalizes the process of recognizing synonyms and internalizing them.

Example 22 *Following the synonyms indicated in example 21, the *-perception of example 1,* $\mathcal{P}_1^*$*, has the following reduced set of connotations:*
$\mathcal{I}_1^* = \{[A\_flat], A, [B\_flat], B, C, [C\_sharp], D, [E\_flat], E, F, [F\_sharp], G\}$.
This perception features internalization of 'well tempered' for the piano environment.

As a matter of fact, synonyms were encountered before, In the context of p-coequalizers (proposition 12). Two p-morphisms can be coequalized if and only if they map every connotation into synonym connotations, and their p-coequalizer simply merges these synonyms. An equivalence relation is also a tool in the proof of that proposition. (Indeed, coequalizers are the categorical recast of equivalence relations.) In the category of perceptions the reduction of the set of connotations, from $\mathcal{I}$ to $\mathcal{I}^*$, can be formalized with a p-coequalizer. Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{P}^* = \langle \mathcal{I}^*, \varrho \rangle$. Define the mapping into the quotient set: $Syn_{\mathcal{P}} : \mathcal{I} \to \mathcal{I}^*$ by $Syn_{\mathcal{P}}(\alpha) = [\alpha]$ (where $[\alpha]$ designates the class of all synonyms of α).

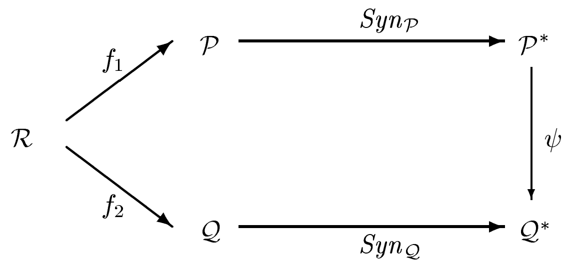

Figure 4: Commutative diagram for proposition 23

Proposition 18 *$Syn_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}^*$ is a p-coequalizer.*

Proof. $Syn_{\mathcal{P}}$ is rigid and onto (epi) by definition. The proposition then follows by proposition 10. □
Example 6 consists of a p-coequalizer that merges synonyms. A general p-coequalizer typically merges only some of the synonyms, performing a *partial merging of synonyms*. proposition 12 can now be rephrased in a stronger form:

Proposition 19 *A p-morphism is a p-coequalizer if and only if all it does is the merging of synonyms.*

Proof. If a p-morphism is a p-coequalizer then it merges synonyms by proposition 12. The proof that a p-morphism which merges synonyms is a p-coequalizer is essentially the same as the proof that a rigid p-epimorphism is a p-coequalizer (second part of the proof of proposition 10). □

Proposition 20 *The reduction of the set of synonyms $Syn_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}^*$ is unique up to isomorphism: If $f : \mathcal{P}^* \to \mathcal{Q}$ is another p-coequalizer then f is a p-isomorphism.*

Proof. It is enough to show that f is one-to-one. Let $\mathcal{P}^* = \langle \mathcal{I}^*, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$. Assume that $f(\alpha) = f(\beta)$. By the rigidity of f it follows that $\alpha \simeq \beta$, hence $\alpha = \beta$ in $\mathcal{I}^*$. f is thus a rigid-epi-mono-morphism: a p-isomorphism. □
$Syn_{\mathcal{P}}$ is thus 'the ultimate p-coequalizer', unique up to isomorphism. Other p-coequalizers typically perform just a partial merging of synonyms.

With the last results it is easy to obtain some basic combinatoric estimates.

Proposition 21 *The universal perception $\mathcal{U}_{\mathcal{E}}$ has a reduced set of connotations.*

Proposition 22 *The universal perception $\mathcal{U}_{\mathcal{E}}$ has the maximal number of connotations for a reduced set of connotations: for all *-perceptions: $|\mathcal{I}^*| \leq 2^{|\mathcal{E}|}$.*

Hence the number of connotations in $\mathcal{P}rc_{\mathcal{E}}$ varies from zero for the initial perception to $2^{|\mathcal{E}|}$ for the universal perception, provided that the set is reduced (i.e. there are no synonyms).

Two perceptions that differ only in the number of synonyms for their connotations are certainly not very different one from the other. This is the essence of the following proposition:

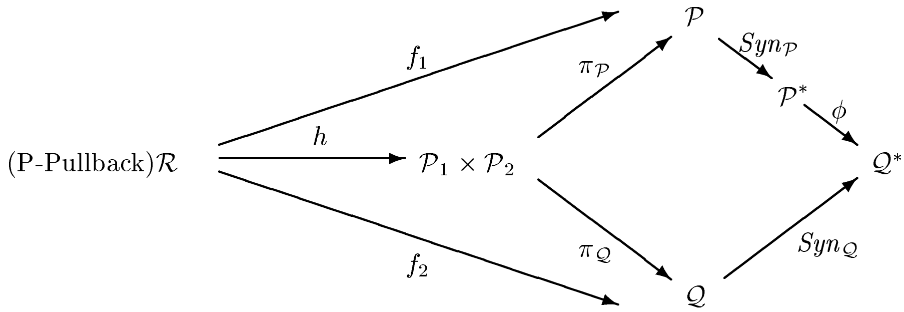

Figure 5: Commutative diagram for necessity part of proposition 23

Proposition 23 *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$, then the existence of a perception $\mathcal{R}$ with two p-coequalizers $f_1 : \mathcal{R} \to \mathcal{P}, f_2 : \mathcal{R} \to \mathcal{Q}$. is necessary and sufficient for the existence of a p-isomorphism $\psi : \mathcal{P}^* \to \mathcal{Q}^*$.*
In that case $f_1 \circ Syn_{\mathcal{P}} \circ \psi = f_2 \circ Syn_{\mathcal{Q}}$ and $\mathcal{R}^*$ is isomorphic to both $\mathcal{P}^*$ and $\mathcal{Q}^*$. (see figure 4).

Proof. Necessity is shown first: (proof illustrated by figure 5): Assume the p-isomorphism $\psi : \mathcal{P}^* \to \mathcal{Q}^*$. We shall define $(f_1 : \mathcal{R} \to \mathcal{P}, f_2 : \mathcal{R} \to \mathcal{Q})$ to be the p-pullback of $(Syn_{\mathcal{P}} \circ \psi, Syn_{\mathcal{Q}})$, and then show that the f's are p-coequalizers. To construct that p-pullback we use the method that was described in figure 3. First take the product perception $\mathcal{P} \times \mathcal{Q}$. The composite of its projections, $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{Q}}$, onto (respectively) $\mathcal{P}$ and $\mathcal{Q}$, with (respectively) $Syn_{\mathcal{P}} \circ \psi$ and $Syn_{\mathcal{Q}}$, yield two p-morphisms, $\pi_{\mathcal{P}} \circ Syn_{\mathcal{P}} \circ \psi, \pi_{\mathcal{Q}} \circ Syn_{\mathcal{Q}} : \mathcal{P} \times \mathcal{Q} \to \mathcal{Q}^*$. We claim that they have a non-empty p-equalizer $h : \mathcal{R} \to \mathcal{P} \times \mathcal{Q}$. ψ and the Syn's are rigid. Hence the domain $\mathcal{R} = \langle \mathcal{I}_R, \varrho_R \rangle$ of the p-equalizer h should be $\mathcal{I}_R = \{(\alpha, \beta) \in \mathcal{P} \times \mathcal{Q} || \forall \mathrm{w} \in \mathcal{E} \quad \varrho(\mathrm{w}, \alpha) = \tau(\mathrm{w}, \beta)\}$, where, for all $\mathrm{w} \in \mathcal{E}$, $\varrho_R(\mathrm{w}, (\alpha, \beta)) = \varrho(\mathrm{w}, \alpha) = \tau(\mathrm{w}, \beta)$. $\mathcal{R}$ is not an empty perception because for every two connotations α of $\mathcal{P}$ and β of $\mathcal{Q}$ there exists a connotation (α, β) in $\mathcal{I}_R$: ψ is a p-isomorphism, so whenever $\psi([\alpha]) = [\beta]$, it follows that $(\alpha, \beta) \in \mathcal{I}_R$. By the construction of pullbacks it follows that $(f_1 = h \circ \pi_{\mathcal{P}}, f_2 = h \circ \pi_{\mathcal{Q}})$ is the p-pullback of $(Syn_{\mathcal{P}} \circ \psi, Syn_{\mathcal{Q}})$, so that $f_1 \circ Syn_{\mathcal{P}} \circ \psi = f_2 \circ Syn_{\mathcal{Q}}$. Moreover, the composites $h \circ \pi$'s are epi. They are rigid as composites of two rigid p-morphisms (a p-equalizer and a projection). These are the p-coequalizers that had to be shown. (Loosely, if two perceptions yield the same *-perception, they are both results of partial mergings of synonyms of a third perception with even more synonyms then both of them. The product of the two perceptions has this property, provided that pairs of 'non-synonyms' are discarded. This third perception will, of course, yield the same *-perception.) To show that $\mathcal{R}^*$ is isomorphic to $\mathcal{Q}^*$ (and hence also to $\mathcal{P}^*$), note that $f_2 \circ Syn_{\mathcal{Q}} : \mathcal{R} \to \mathcal{Q}^*$ is a composite of two p-coequalizers and is thus a p-coequalizer itself. Its codomain, $\mathcal{Q}^*$, is a *-perception, and is thus isomorphic to $\mathcal{R}^*$.

Sufficiency is shown now: (proof illustrated by figure 6). Assume that there exists a perception $\mathcal{R}$ with two p-coequalizers $f_1 : \mathcal{R} \to \mathcal{P}, f_2 : \mathcal{R} \to \mathcal{Q}$. We need to show a p-isomorphism $\psi : \mathcal{P}^* \to \mathcal{Q}^*$ such that $f_1 \circ Syn_{\mathcal{P}} \circ \psi = f_2 \circ Syn_{\mathcal{Q}}$. Since f_1 is a p-coequalizer, it has a rigid (choice) left inverse $g : \mathcal{P} \to \mathcal{R}$. Define $\forall \alpha \in \mathcal{I} \psi([\alpha]) = g \circ f_2 \circ Syn_{\mathcal{P}}(\alpha)$. ψ is rigid because it is a composite of rigid p-morphisms. ψ is epi by the following considerations: Let $\beta \in \mathcal{Q}$, and consider $[\beta] \in \mathcal{Q}^*$. It has to be shown that $[\beta] \in \psi(\mathcal{P}^*)$. f_2 is epi (as a coequalizer),

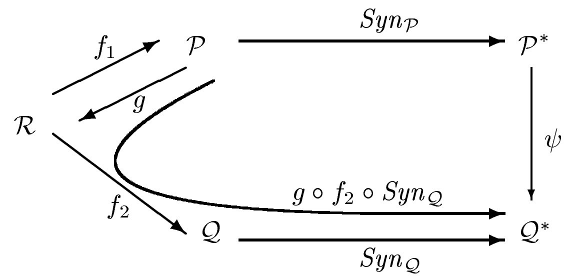

Figure 6: Commutative diagram for sufficiency part of proposition 23

hence there exists $\beta' \in \mathcal{R}$ such that $f_2(\beta') = \beta$. By the definition of g as a choice inverse of the p-coequalizer f_1, there exists a synonym β'' of β' such that $\beta'' \in g(\mathcal{P})$. It follows, by defintion of ψ, that $\psi([f_1(\beta'')] = g \circ f_2 \circ Syn_{\mathcal{P}}(f_1(\beta'')) = [\beta]$. ψ is thus both rigid and epi - a p-coequalizer. Since both its domain and codomain are *-perceptions, it is also mono and hence a p-isomorphism. By its definition $f_1 \circ Syn_{\mathcal{P}} \circ \psi = f_2 \circ Syn_{\mathcal{Q}}$. □
It follows that, in the context of $\mathcal{P}rc_{\mathcal{E}}$, the familiar categorical expression 'unique up to isomorphism' will often be replaced by the somewhat weaker 'unique up to synonyms', and the proofs of similarity will consist of providing suitable p-coequalizers. A special case is when one of the two perception always has less synonyms then the other:

Example 23 *Let $\mathcal{P} = \langle \mathcal{I}, \varrho \rangle$ and $\mathcal{Q} = \langle \mathcal{J}, \tau \rangle$, if there exists a p-coequalizer $f : \mathcal{P} \to \mathcal{Q}$ then $\mathcal{P}^*$ and $\mathcal{Q}^*$ are isomorphic. ($\mathcal{P}$ replaces $\mathcal{R}$ in proposition 23.). An example private case is provided by $\mathcal{P}_{1a}$ of example 2 and $\mathcal{P}_2$ of example 3 that are similar up to synonyms, and the required p-coequalizer is h of example 6.*

8.3 Refining Coproduct Joint Perceptions

Consider a 'maximal trust' joint perception using a coproduct as in proposition 15. In spite of the differences between the perceptions that are being joined, there could well be connotations from different perceptions that are essentially the same. These would be synonyms in the coproduct perception, and they can be merged with a suitable p-coequalizer.

Example 24 *In a coproduct perception $\mathcal{P}_{1a} \oplus \mathcal{P}_2$ (of examples 2 and 3), the following are synonyms: $A \simeq La$, $B \simeq Si$, $C \simeq Do$, $D \simeq Re$, $E \simeq Mi$, $F \simeq Fa$, $G \simeq Sol$. If these are merged, then not only both notation systems are internalized, but also these similarities between them.*

Categorically, a pushout is constructed out of the coproduct and a coequalizer, as illustrated by figure 7. The intuitive idea behind p-pushouts is thus to enhance partnership in coproduct perceptions by merging synonyms, using the coequalizer h in the diagram. Figuratively speaking, why have 'your connotation' and 'my connotation' when they are essentially the same, and could be generalized into one single common connotation. This cognitive process does not need to be performed for *all* such connotations, so that it is only a lax analog to set union.

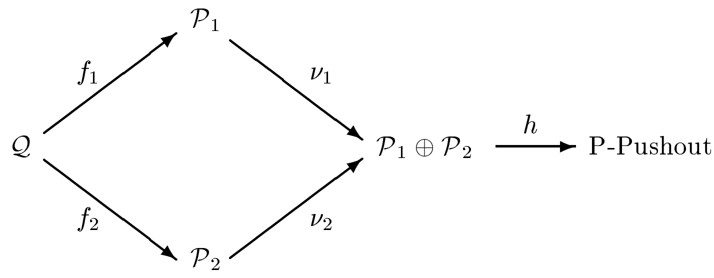

Figure 7: Maximal Trust Joint Perception with Partnership: P-Pushout

To apply the theory to a specific case of maximal trust joint perception: Let the $f_\imath$'s of figure 7 point (for every connotation of a trivially chosen $\mathcal{Q}$), at similar connotations from $\mathcal{P}_1$ and $\mathcal{P}_2$, that would yield synonyms in the coproduct. Not all such synonyms have to be merged. One is free to define $f_\imath$'s that provide the desired partnership. Consider the p-pushout of f_1, f_2, defined by the p-coequalizer of $f_1 \circ \nu_1$ and $f_2 \circ \nu_2$. It yields the coproduct $\mathcal{P}_1 \oplus \mathcal{P}_2$ with the desired merging of synonyms.

Example 25 *Formalizing commonsense connotations: Assume that certain pitch connotations were shared by many musical notation systems (varying across time and cultures). This would bear information about 'shared', 'commonsense', musical perception. Consider a family $\{\mathcal{P}_\imath\}_{\imath=1}^n$ of perceptions. Using the pushout formalism described above (and the notation from proposition 15), one could merge sets of n synonyms of the form $\{(\alpha, i)\}_{\imath=1}^n$. Namely: there is at least one synonym in every single member of the family of participating perceptions. Such sets of synonyms, if they exist, represent connotations that are shared by all participating perceptions (sharing is in essence, the internal representations, the α's, may vary). This may open a rigorous way to go about formalizing meanings that are commonsense to a large family of perceptions. Indeed:* shared $\approx$ common, *and* connotation $\approx$ sense.

9 Further Applications of Artificial Perception Theory

More powerful tools of category theory, such as natural transformations and free functors are needed for the study of *Boolean Perceptions.* They are discussed in [3], and a short summary appears in [2]. Boolean perceptions are perceptions of $\mathcal{E}$ with a boolean algebraic structure on their sets of connotations. They form a subcategory. Their perception predicates can be characterized both externally with categorical boolean tools and internally with three valued logic. This solves, among other things, a problem that was mentioned in this study (in the context of example p-morphisms): we would like to be able to state, for example, that a piano key has the connotation *black* whenever it has *either one* of the *flat/sharp* connotations. Boolean perceptions provide tools for formalizing quite a few artificial cognitive processes. Two free constructions from the category of perceptions (or some subcategory) into the boolean subcategory formalize reasonable ways to go about producing a meaningful cognitive image of the environment from every perception. This communicates to the higher level reasoning modules all the information which is extractable from raw perception, set in an explicit logical form. Artificial perception theory is thus also able to formalize a bridge that integrates perception with higher reasoning (i.e. problem solving, decision making etc). Casting that bridge as a free functor and a natural transformation exploits more advanced tools of category theory. Beyond the expressive rigorous tools, the categorical infrastructure warrants a unified theory: Other cognitive capabilities, such as communication between perceptions, and the cognitive capabilities that were presented in this paper, will be preserved, and even enhanced, during any further processes that will be formalized by this infrastructure.

10 Summary

The category of perceptions provides a rigorous mathematical environment within which one may scrutinize artificial perception and cognitive processes related to that perception. AI concepts and ideas are meticulously cast in mathematical form and are hence available for a detailed scientific treatment. In the present study we formalized, among other things, transition and comparison between different perceptions, improving and completing an agent's perceptual grasp, and several forms of joint perception. These, as well as further applications, constitute a promising hope that the mathematical formalization is worth the effort for AI.

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