Perceive This as That

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Abstract

This paper formalizes and analyzes cognitive transitions between artificial perceptions that consist of an analogical or metaphorical transference of perception. The formalization is performed within a mathematical framework that has been used before to formalize other aspects of artificial perception and cognition. The mathematical infrastructure consists of a basic category of 'artificial perceptions'. Each 'perception' consists of a set of 'world elements', a set of 'connotations', and a three valued (true, false, undefined) predicative connection between the two sets. 'Perception morphisms' describe structure preserving paths between perceptions. Quite a few artificial cognitive processes can be viewed and formalized as perception morphisms or as other categorical constructs. We show here how analogical transitions can be formalized in a similar way. A factorization of every analogical transition is shown to formalize metaphorical perceptions that are inspired by the analogy. It is further shown how structural aspects of 'better' analogies and metaphors can be captured and evaluated by the same categorical setting, as well as generalizations that emerge from analogies. The results of this study are then embedded in the existing mathematical formalization of other artificial cognitive processes within the same premises. A fallout of the rigorous unified mathematical theory is that structured analogies and metaphors share common formal aspects with other perceptually acute cognitive processes.

1 Introduction

Cognitive transitions between settings that are perceived as analogical, in some sense, constitute a salient natural activity of human intelligence. They have been recognized and studied since antiquity. In an AI context the fruitfulness of analogies should typically depend on whether any testable consequences could be deduced from them. Analogizing could sometimes be useful as a cognitive tool for perceptive artificial intelligent agents.

- Analogies may explicate unfamiliar environments and situations in terms of more familiar ones. For example, an intelligent artifact that has been trained to perceive a bookstore environment and interact with (e.g. 'shop' in) this environment, can be efficiently taught to do the same in a music store, with the necessary analogs being drawn. Likewise, an intelligent artifact that has been trained to perceive music and interact with (e.g. 'play on') the keyboard of a piano can be efficiently taught to do the same with another keyboard instrument, the necessary analogs being drawn.

- In rational thinking, analogies may be used to suggest hypotheses about a more general rule or setting. For example, an analogy between a bookstore environment and a music store environment may suggest to an observant intelligent artifact general patterns of behavior of agents selling and/or shopping in store environments. Consequences could be deduced for general store environments. Likewise, the analogy between a piano and another keyboard instrument may suggest to an observing intelligence general patterns of music playing. Consequences could be deduced for keyboard instruments in general.

- Linguistic translations constitute examples of analogies. One may look at a language as an environment that consists of perceptible audio/visual elements (words, phrases, etc.). A translation is thus a cognitive transition to the respective environment of another language. Analogs are typically being drawn between linguistic elements that have the same semantic properties, but they can also preserve (or consistently map) the audio properties of the linguistic element, (such as measure and rhyme in translations of poetry), cultural insinuations [51], or even visual properties of the linguistic element, if there is, for example, a perceptual sensitive transference of font.

- In their interaction with human agents, intelligent artifacts might need to analogize when they follow a human line of reasoning (e.g. in the process of expert knowledge acquisition). Metonymies, metaphors, tropes and a variety of other expressive tools that are based on analogies have pervaded human cognitive processes to a point that it would be hard to avoid them when communicating with artificial systems. Lakoff [41, 40, 39], for example, argues that language is tropological, and that its usage is typically inspired by analogy to bodily experiences, as in '*arriving* at a solution' or '*seeing* a point'.

A mathematical theory of artificial perceptions is proposed as a framework for the formalization of cognitive transitions between settings that are analogical in some perceptible sense. It is shown that the theory can naturally formalize analogy-making, often a useful cognitive activity.

The body of research about analogies in thought, science, language, and other cognitive domains and activities is so vast that it is hard to say anything new, that has not already been said before, about analogies. This paper neither competes with existing theories (some of them will be briefly sketched in the next section), nor does it take sides in debates among various philosophical stances (e.g. traditional comparison theories, interactionist theories, and others). The force of a categorical approach is in avoiding over determination [48], and hence the paper will also not deal with various semantic distinctions between analogies, similes, models and other related phenomena. The novelty and the goal of the proposed approach is the introduction of a rigorous mathematical categorical framework, where no such framework already exists. In addition to meticulous mathematical rigor, this approach opens the way for the integration of analogy making with other cognitive processes that are based on the same mathematical formalism.

2 Background and Related Works

2.1 Models of Analogy-Making

Analogy making has always been a natural human activity, and biblical parables (typically euphemistic in nature) constitute early examples. Likewise, the recognition and the study of this multifaceted phenomenon date back to antiquity. Philosophers and scientists have continuously employed and studied this versatile cognitive process. Artificial intelligence and cognitive science have lately aroused a renewed interest in analogies from their perspectives. Thagard [55] counts the study of analogies as one of the fundamental theoretical approaches to mental representation.

Making and following analogies in an artificial system necessitates a meticulous formulation of greater rigor than the verbal descriptions that had been employed before AI. One is now faced not only with the goal of *recognition* and critical analysis of analogies and metaphors, but also with a new potential task of the *generation* and creative synthesis of analogies and metaphors by an artificial intelligent system. One of the first attempts in this direction was Thomas Evans' program ANALOGY, described in [49], that was designed to solve visual geometric analogy problems in which it had to pick one of five possible solutions to problems of the form 'A is to B as C is to ?'.

Gentner [25] surveys the history of the study of analogy as a fruitful interdisciplinary convergence between psychology and AI, with significant influences from history of science, philosophy, and linguistics. She models the use of analogy in learning and reasoning by subprocesses: retrieval, mapping and structural alignment, evaluation, abstraction, re-representation and adaptation. Holyoak and Thagard [33] present a theory of analogy that covers evolution of thinking in animals and in children. They consider its implications for cognitive science in general, with examples from various domains of human activity. [56] is an interdisciplinary volume that explores research and theory of similarity and analogical reasoning from psychological, educational, and computational perspectives. The computational perspectives deal with problems encountered in simulating analogical processing in reasoning and problem solving. The computational model SME in [20, 24] emphasizes the role and import of structure mapping in analogies: analogy is viewed as a mapping between structured representations.

Indurkhya [35, 36] proposes a view of metaphor as change of representation. A key to his approach is the distinction between the concrete environment (which he calls SMD: sensory-motor-data) and highly abstract concept networks, and the process of interaction between these two levels that tries to preserve the autonomous structures of both, while making the concepts correspond to the stimuli in an SMD. This key issue, as well as other elements of his approach, are shared in some sense with the formalism proposed in this paper. The overlaps will be pointed out at the relevant places.

Mitchell [50] and French [22] describe two computer models of analogy-making that model the complex interaction between perception and concepts that underlies the creation of analogies. Hofstadter [31] chronicles efforts, including the two last models as major examples, to develop computer models that provide insights into discovery, creation, and analogical thought. A key idea that emerges from the last three cited works is that high-level perception of situations, of patterns, and patterns of patterns is at the roots of the cognitive processes involved.

In the machine learning literature [44] analogical inference comes under the epithet 'case-based reasoning'. Kolodner [38] shows the development of applications of analogical reasoning in the AI context of case-based reasoning. A solution to a problem can be formed by adapting a previously solved problem. One needs to retrieve a potentially relevant case from memory and establish the correspondences between the cases. One may then abstract the common properties of the two cases, providing a schema that can be used for further problem solving. This is essentially the same as Gentner's subprocess of abstraction that was mentioned above. A process in this spirit will be formalized here by a *pullback* categorical construction.

2.2 Human Cognition and AI models

Most of the research about analogies that was cited above, including the computational models, has typically had the goal of understanding, illuminating, and imitating human cognitive processes. For psychologists, philosophers, and linguists, the issue of *how* an AI artifact produces or follows an analogy is often inseparable from the fact that the artifact *can* produce or follow an analogy. Human intelligence has indeed been a source of inspiration for research in AI, and this study is no exception. However, this work is not committed to being empirically adequate from a human psychological point of view.

A model is meant to represent something that can not be directly, or easily, observed (e.g. an atom, or processes in a human mind) by another system which is more familiar or more easily effected, and whose workings are supposed analogous to that of the first, at least *in some respects*. There is selective emphasis on the features that overlap, while ignoring other features. One might model the behavior of sound waves upon that of waves in water, or the behavior of a gas upon that of a volume containing moving billiard balls. However, molecules are not billiard balls, buildings are not made of cardboard, and their miniature models are not made of concrete.

A mathematical model is typically a system of definitions, assumptions, and equations that are set up to discuss real world phenomena that are met by a physicist, an economist, an engineer, and maybe also a cognitive scientist. Geometry emerged as a model of the physical space in ancient Greece. From the beginning of the 17th century, analysis developed to model motion and equilibrium of physical bodies. Mathematical probability models problems of gambling and human chance, as well as statistical theories of mechanics and thermodynamics. There are, of course, more examples. To apply the mathematics it is typically necessary to develop an abstract mathematical system, that approximately corresponds to the real world phenomenon. This may involve making assumptions and simplifications. If an AI artifact is programmed according to a formal schema (e.g. the one that is proposed here) to perceive the world and analogize, its sensory-motor-neural apparatus, that interacts with its environment, will not consist of living cells. Its workings will be similar to human cognitive workings only in some respects, and drastically different in others. It will be argued, however, that there is enough parallelism to call this *a model* of artificial perceptive cognition.

A model is a form of analogy. There are hence two distinct levels of analogy present in this study. There is the general phenomenon of analogy making that is being studied, and a specific analogy that is being drawn between human cognitive processes and constructs of the proposed formal schema. Some researchers resist to the legitimacy of the latter. Sloman [54] argues, against those whom he calls 'doubters and fearers', in favor of the idea that AI artifacts may have cognition (and even emotions). The debates about the possibility to model human cognitive processes by formal artificial systems are, in a certain sense, analogous to some old theological debates about attributing similar terms to man and to God. Philosophers of religion and theologians have recognized the importance of analogy for the exposition of doctrine. The medieval theologian Thomas Aquinas claimed that *an attribute that is applied both to man and to God is not used in a univocal sense (in the same sense), nor is it used in an equivocal sense (totally different sense), but it is used in an analogical sense (a similar sense). Though differences are great between man and God, there is enough similarity that one is justified in using the same terms to apply to both. When the same term is used, however, it must be understood that it is used in an analogical sense when it is applied to God.* Hence, yet another analogy that is suggested here is drawn between, on one hand, the analogy that was made by Thomas Aquinas and, on the other hand, the analogy between AI systems and the cognitive systems of their human creators[1]. When terms such as 'sensory-motor-neural' apparatus are applied to machines, they are used in an *analogical* sense. Analogy is ubiquitous, which is why it ought to be studied.

We propose a mathematical categorical formalization of artificial perceptions as a 'unified standard' for the modeling of perception-related cognitive processes, analogies and metaphors being salient examples. Ideas and concepts from research about human cognitive processes provide pre-theoretical intuitions. They are being systematized and modeled by mathematical context free premises. Constructs and results are inferred from the formal premises applying methods and results from category theory, Boolean algebra and Lukasiewicz's (three valued, non-monotonic) logic [46, 45]. The suitability of these tools to model human cognition could, indeed, be debated. They are not suggested as the human processes themselves, but rather as parallels that can be applied in a formal or artificial context. Whenever results are reached, it is possible to continue the dialogue with theories of human cognition, and examine the results against these theories. This will be done as we proceed with the constructions. It will be argued that the resemblances noted bear relevantly on analogy-making, and that the parallelism extends beyond the definitions. Starting from minimal definitions, more constructs are generated, along with propositions and theorems that pertain to these constructs. If one manages to obtain constructs and properties that would not have been obvious had they been asserted at the outset, and they resemble certain aspects of cognition, then this should provide additional support for the suitability of the proposed schema.

Some foundational intuitions that the proposed theory shares with studies of human analogy-making and tries to capture and systematize, are:

- High level perception (in parallel) of sensory-motor-neural impressions of cohesive wholes in the environment provides basis for cognitive processes such as analogy-making.

[1]Figuring out the details of this scandalous humanistic analogy is left to the reader.

- 'Good' analogies are based on *structured* mappings of perceptual constituents, of patterns of these constituents, and of patterns of patterns.
- An important aspect of analogy is a process of abstraction of a superordinate schema from the analogs.

More overlaps will be pointed out at relevant points, as the presentation of the formalism proceeds.

2.3 Category Theory as a Tool

Category theory emphasis on mappings between structures provides a natural motivation to apply categorical tools for a formalization of analogies. Kant observed ([14, p.66]) that cognition by analogy does not signify an imperfect similarity of two things, but a *perfect similarity of relations between two quite dissimilar things.* Contemporary works about analogies and metaphors, as mentioned above, share observations that a good analogy should be based on structured mappings. The issue of similarity between structures is fundamental in mathematical category theory, started by Eilenberg and Maclane in [19]. Some later texts about category theory are [29, 47], and even more recently [52, 57, 6, 16, 12, 7, 43]. (Kant was, indeed, the one who introduced the philosophical idea of categorization in his 18th century statement '*Grounding for the Metaphysics of Morals*' [37].)

The term *structured mapping* appears frequently in studies about analogy. Some researchers actually use variants of the terms *morphisms, isomorphisms, homomorphisms*: Black [10], Indurkhya [35, 36], Holland and Quinn [32], and possibly others. In most cases, however, their usage is rather loose. Indurkhya's *cognitive models* are perhaps closest to the mathematical formalism that is proposed in this study. The advantages of the full category theoretical setting that is proposed here is the possibility to rigorously integrate quite a few perceptual cognitive processes into one unified theory, using, for example, composition of morphisms and commutative diagrams. Category theory also provides other, more sophisticated, tools such as (but not restricted to) natural transformations. These tools have been used in other papers [5, 4, 3, 2] that build on the same formalism. They will be discussed later.

In the general context of more formalizations of other cognitive processes, in addition to analogies, mathematical category theory seems to be a natural candidate as well. Lawvere argues [42] that 'Even within mathematical experience, only... [category]... theory has approximated a *particular* model of the general, sufficient as a foundation for a *general* account of all particulars'. Following this theme, category theory seems suitable for purposes of AI. This science tries, in a sense, to approximate intelligence by creating particular models of artificial intelligence as well as foundations for a general account of such intelligence. In particular, the mathematical categorization provides a setting for the approximation of artificial cognitive processes by describing particular perceptual cognitive processes in particular intelligent artificial settings, as well as foundations for a general account of intelligent artificial perceptive cognition. Marrying mathematical category theory with cognitive studies is also proposed by Magnan and Reyes [48], who suggest that 'universals of the mind may be expressed by means of universal properties in the theory of categories', and that categorical universals constitute 'blueprints' of the mind.

Further motivation for the application of mathematical categorical methods for purposes of formalizing artificial perceptions and related cognitive processes has already been argued in [5, 4] that introduce and discuss the mathematical premises upon which this study is based. They also provide a summary of some representative AI research that employs category theoretical tools. Barwise and Seligman's [8] research can be added to that list. They have also found that the basic perspective of category theory could be quite helpful as a guide in developing their theory of information flow within a connected distribution system.

2.4 The Category and the Theory of Artificial Perceptions

Following the motivations to apply category theory to artificial perceptual cognitive processes, a category of artificial perceptions was introduced in [5]. Perceptions were defined as a domain of mathematical discourse, where different perceptions represented different objects of the category. Morphisms described structure preserving paths between perceptions. This provided tools of scrutiny for dealing with all perceptions. Structural similarities among perceptions could be studied, yet leaving ample room for differences and variety. The categorical setting served as premises for a mathematical theory. Its technical basis will be presented in section 3 below.

Among the applications proposed in [5] were categorical products, coproducts, pullbacks and pushouts that were applied to formalize joint representations with varying degrees of partnership and trust. The theory was further applied for more cognitive processes in [4], where free functors and natural transformations were applied to formalize reasonable ways to go about producing a meaningful cognitive representation of the environment from every perception. In the present study the theory will be incremented by showing how analogical transitions can also be captured by categorical constructs. For this purpose the basic definitions will be extended:

- In the cited works the environment was fixed. The definitions allowed for a variety of environments, however the generic representative (sub)category that was studied consisted of perceptions of a fixed environment, with morphisms as paths between various subjective representations and ontologies of that fixed environment. In this paper this family of (sub)categories, each with a fixed environment, will be integrated into one category. Morphisms in the unified category will also enable transitions between perceptions that may consist of transferences and manipulations of environments as well as transferences and changes of representations.

- Structure and structure preservations in the cited works applied to the representation. In this paper structural elements are introduced into the environment as well, and the concept of structure preservation applies also to transitions between perceptions of different environments.

It is noted, however, that the main import of the present paper is not in these technical extensions to former works, but rather in highlighting common themes: The same mathematical setting can be repeatedly employed to formalize a variety of cognitive processes. This means that:

- The premises do capture basic issues that are relevant to many cognitive processes, suggesting a theoretical standard.
- Whenever a construct or a result is repeatedly applied to more than one cognitive capability or process, then this seems to highlight a central cognitive issue. In that case specific references (including page numbers) to any previous mathematical treatment, are provided.
- The analysis of a variety of perceptual cognitive processes in terms of a relatively small number of primitives predicts the possibility of tidily structured implementations with a reduced component set. Context free modules may perhaps be reusable for different cognitive processes.

Application of mathematical methods for purposes of AI is not new [9, 18]. Mathematical rigor is essential in an artificial context, because architectural requirements need to be meticulously formalized in order to be implemented. The advantages of mathematical formalizations as analyzed, for example, in the introduction to [17] include clarity, precision, versatility, generalizability, testability, allowance to model complex phenomena that are far too complex to be grasped by a verbal description, and allowance to use results of a well developed science.

3 The Categorical Premises

The essentials of the mathematical premises are presented first. (They extend those of [5, 4] by allowing transitions between perceptions of different environments.) The abstract idea of a perception is postulated as a mathematical construct which relates between phenomena outside the artificial agent, a set of *world elements* and reflections which are internal to the artificial agent, a set of *connotations*.

Definition 1 *A* Perception *is a three-tuple* $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ *where* $\mathcal{E}$ *and* $\mathcal{I}$ *are finite, disjoint sets, and* ϱ *is a 3-valued predicate* $\varrho : \mathcal{E} \times \mathcal{I} \to \{t, f, u\}$.

The set $\mathcal{E}$ represents the environment which the machine perceives. Anything which exists independent of the perception itself, and could perhaps be discerned by it, could be a legitimate element of $\mathcal{E}$, and hence a *world element* (*w-element* for short). Possible example w-elements are a face, a light blow of wind, the shadow of a smile, a slight shivering of voice, a tinge of smell or taste or color, etc. Not every sensory-motor-neural mechanism is able to discern every such outside phenomenon, and even if it does, it may not be able to represent it internally, or attach 'universally conventional' connotations to them. Different perceptions might break the same reality into different parts that are regarded as cohesive wholes. Although the external environment has an objective existence, its organization into w-elements is subjective. It is impossible to separate perception from the environment to which it relates. The essential role that interaction with an environment plays in intelligence has been recognized at least since Turing, and recently solicited by others, like Clark [15], Wells [58], and Allen [1], to name just a few. The idea of a cognitive *supraindividual* that includes its environment was also proposed by Hutchins [34]. It is also a pre-theoretical intuition of this study that intelligence develops relative to a given, embodied, sensory-motor-neural apparatus on one hand, combined with a given environment on the other hand. They provide the 'bottom-up' aspect of perceptions: from stimuli to representations.

The set $\mathcal{I}$ stands for the internal representation of w-elements. Its elements have a subjective existence dependent on the machine. Anything which may be stored and manipulated in the machine (words, symbols, icons, etc.) could be a legitimate element of $\mathcal{I}$, and hence a *connotation*. This provides the 'top-down' aspect of perceptions: from the representations to the chunking and the structuring that they impose on environments. Indurkhya [35, 36] argues convincingly for the importance of the interaction between an autonomous *Reality*[2] which presents itself in the form of 'raw material' sensory stimuli, and its internal chunking and conceptualization.

The three-valued predicate ϱ is the *Perception Predicate* (*p-predicate* for short) which relates w-elements and connotations. The terminology for the various ϱ values is the following: If $\varrho(\mathrm{w}, \alpha) = t$ then w *has connotation* α, if $\varrho(\mathrm{w}, \alpha) = f$ then w *lacks connotation* α, and if $\varrho(\mathrm{w}, \alpha) = u$ then w *may either have or lack this connotation*. This undefined value might eventually become defined, but right now it is not.

Every perception has its own set of w-elements, its own set of connotations, and its own predicative correspondence between the sets. They are given once the instance perception is fixed, in very much the same way that the details of a cup are accessible once perception relates to a specific instance of a cup from the category of cups. Actual sets $\mathcal{E}$ and $\mathcal{I}$, and the values of the p-predicate, once given, provide a definition of a particular perception. This captures the intuition that perceptions and sensations are innate to agents: their gestalt perceptions, mental imagery, neural-sensory-motor apparatus, function, internal organization etc. Likewise, the issue of *why* the p-predicate has any one of the three values at a certain point simply warrants no discussion: impressions are not necessarily explained.

The mathematical objects stand for embodied perceptions or perceptual states. They are high-level in the sense that they reside higher than pixels on the screen or waveforms of sound. They are object centered and therefore happen at and above the level of recognition of cohesive wholes, where meaning and conscious cognizance begin to play a role. (Connectionism has been suggested as a candidate for the mechanism that underlies this level [28].)

Perceptions vary across agents, situations, etc. The flexible dynamic flow between different high-level perceptions is formalized by perception morphisms between them:

Definition 2 *Let* $\mathcal{P}_1 = \langle \mathcal{E}_1, \mathcal{I}_1, \varrho_1 \rangle$ *and* $\mathcal{P}_2 = \langle \mathcal{E}_1, \mathcal{I}_2, \varrho_2 \rangle$ *be two perceptions.* $h : \mathcal{P}_1 \to \mathcal{P}_2$ *is a* Perception Morphism *(*p-morphism *for short) if* h *defines the following set mappings:* $h : \mathcal{E}_1 \to \mathcal{E}_2$*,* $h : \mathcal{I}_1 \to \mathcal{I}_2$*, and* No–Blur *is the structure preservation condition: For all* $\mathrm{w} \in \mathcal{E}_1$*, and for all* $\alpha \in \mathcal{I}_1$*, whenever* $\varrho_1(\mathrm{w}, \alpha) \neq u$ *then* $\varrho_2(h(\mathrm{w}), h(\alpha)) = \varrho_1(\mathrm{w}, \alpha)$*.*

This definition is an extension of the definition in [5, 4], which defined arrows between perceptions with the *same outside environment*: $\mathcal{E}_1 = \mathcal{E}_2$, and $h : \mathcal{E}_1 \to \mathcal{E}_2$ was the identity mapping.

[2] Capitalized in the reference.

Remark 1 *The definition does* not *imply that both perceptions, $\mathcal{P}_1$ and $\mathcal{P}_2$, exist prior to the transition, and that the p-morphism comes later. There are cases where the transition is* creative *in the sense that either one of the two perceptions gives rise to the p-morphism, actually creating the other perception. (A path between perceptions can be traversed backwards as well, reversing the direction of the arrow. This can be technically effected in the mirror category, as explained in [5, p.278]). Example creative transitions will be mentioned later. This issue is relevant to analogizing because, as Indurkhya [35, 36] argues, some analogies actually* create *similarities rather than just find pre-existing ones.*

Rigid p-morphisms preserve the structure of perception in a rigid fashion:

Definition 3 *h is a* Rigid p-morphism *if, for all world elements $w \in \mathcal{E}_1$, and for all connotations $\alpha \in \mathcal{I}_1$, $\varrho_1(w,\alpha) = \varrho_2(h(w), h(\alpha))$.*

Composition of p-morphisms and the identity p-morphism are defined at the level of set mappings, and it follows that:

Theorem 1 *Perceptions with p-morphisms make a mathematical category, designated $\mathcal{P}rc$.*

The proof is essentially the same as in [5, p.274-275]. This provides a well developed mathematical infrastructure for a 'theory of artificial perceptions', in the same manner as the category of groups is the basis for group theory.

In [5, p.276-277] and in [4, p.199-201] example p-morphisms have already been applied to formalize cognitive processes such as communication and interpretation between different perceptions *of the same environment*. The mathematical properties of the p-morphism (isomorphism, one to one, many to one, onto, impossible etc.) carry meticulous information about how close these perceptions are, and pinpoints differences between them. This categorical framework has been shown to neatly capture other cognitive processes as well. In particular, introduction of classification and mental organization cognitive processes into this sense perception framework is formalized by free endofunctors into the subcategory of *Boolean perceptions*, where sets of connotations are closed under Boolean operations, namely the $\mathcal{I}$'s are Boolean algebras[3]. These Boolean perceptions are studied in [4, p.207-212]. Various cognitive transitions into such perceptions, using free endofunctors, are formalized and studied as well. It is, indeed, natural for cognitive, intelligent, artifacts to sense and perceive Boolean combinations of connotations, and to apply the associated taxonomies. If w is, for instance, a signpost, then a higher-level Boolean perception $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ could perceive that $\varrho(w, \neg small) = f$, $\varrho(w, glittering \wedge ringing) = t$, $\varrho(w, tasty \vee smelly \vee sticky) = u$. The theory provides [4, p.210-211] categorical basis for a deductive apparatus for the computation of the three-valued p-predicate for Boolean combinations of connotations, and it will be employed later for analogizing.

A substantial part of [4] is dedicated to the construction of the p-morphisms that capture high-level representation formation: Starting from basic sensory-motor-neural perceptions and simple representations, organize and shape structured representations of labeled impressions that can be further used for higher-level cognitive processes (i.e. problem solving, decision making, planning, etc.).

[3]Barwise and Seligman also introduce Boolean operations and classification into their theory of information flow [8].

These p-morphisms map basic represented impressions into their Boolean closures, producing logical, cognitive, representations [4]. Two canonical free constructions from the category of perceptions (or some subcategory) into the Boolean subcategory formalize reasonable ways to go about producing a meaningful cognitive image of the environment from every perception. One is general [4, p.212-215], the other [4, p.223-230] is more 'perceptually acute', and it answers to a criterion of mathematical completeness and features a categorical fixed point. Connections between analogies and these processes will be analyzed later.

The theory thus provides embodied cognitive structures that are grounded in authentic sensory-motor-neural impressions and basic representations of actual environments. On the other hand, they are interpretable as logical formulas, the dominant view in AI being that the knowledge content of high level artificial reasoning processes ought to be represented by data structures with this property [23].

In the present study the theory is incremented by showing how analogical transitions can also be captured by categorical constructs. Natural primitive constituents for perceptive analogizing between environments are provided by connotations that are discerned by a creative intelligence. Following an acute analysis of the environment, an observant perception may then proceed to identify a similar recomposition of (mappings of) these constituent elements in other environments, providing for an analogy. In the following sections the process is formally analyzed within the mathematical premises provided above.

In later sections the newly formalized processes are fused into the existing collection of cognitive processes within these premises. The unified theory provides extensions of additional pre-theoretical conceptions, so that the mathematical formalism seems to be useful for AI: A single context free theory integrates the representation of high-level perceptions and a variety of cognitive manipulations.

4 An Example Analogy

Science teachers have traditionally used a camera as an analog to describe the eye [30, 26]. Let us first define a perception of a camera, $\mathcal{P}_{\mathrm{Cam}}$, and a perception of an eye, $\mathcal{P}_{\mathrm{Eye}}$, and then define a p-morphism between the two[5]. Perceptions can be conveniently described by a *Perception Matrix*, where lines represent w-elements, columns represent connotations, and entries consist of the p-predicate value for the corresponding coordinates. In our simple example (table 1) the sets of connotations of $\mathcal{P}_{\mathrm{Cam}}$ and $\mathcal{P}_{\mathrm{Eye}}$ are the same, so that columns are shared. The p-morphism $h : \mathcal{P}_{\mathrm{Cam}} \to \mathcal{P}_{\mathrm{Eye}}$ is based on the identity mapping of connotations, and $h : \mathcal{E}_{\mathrm{Cam}} \to \mathcal{E}_{\mathrm{Eye}}$ is described by letting w and $h(w)$ share lines: h(outside actions)=brain stem reflexes, h(shutter)=iris, h(aperture)=pupil, h(film)=retina, h(camera lens)=cornea, h(lens-film distance)=eye lens+muscle. It is easy to see

[4]These are *creative* p-morphisms in the sense of remark 1: the codomain perception does not exist prior to the transition, but rather the p-morphism transition defines the codomain perception.

[5]Following remark 1, this is not necessarily the order in which things occur. A very creative intelligence might have conceived of a camera by analogy to an eye. In that case the p-morphism would have been conceived to create its domain.

Table 1: Camera Perception and Eye Perception with Analogy

$\mathcal{I}_{\text{Cam}}=\mathcal{I}_{\text{Eye}}$: $\mathcal{E}_{\text{Cam}}$ / $\mathcal{E}_{\text{Eye}}$	*regulate light*	*light sensitive*	*adjust focus*	*refract*	*dynamic change*	*control of other object*
outside actions / brain stem reflexes	t/t	u/u	t/t	u/u	t/t	t/t
shutter / iris	t/t	f/f	f/f	f/f	t/t	t/t
aperture / pupil	t/t	f/f	f/f	f/f	t/t	f/f
film / retina	f/f	t/t	f/f	f/f	f/f	f/f
camera lens / cornea	f/f	f/f	f/f	t/t	f/f	f/f
lens-film distance / eye lens+ muscle	f/f	u/f	t/t	u/t	t/t	f/f

that h is no–blur as required by definition 2. Since h is the identity on connotations, the analogy is *literal*: the similarity between w and $h(w)$ is straight forward, connotation for connotation. The mapping is one-to-one but not rigid (definition 3): *light sensitivity* and *refraction* have no meaning (i.e. undefined) when applied to lens-film distance, but they are defined (f and t, respectively) when applied to eye lens. (see table 1, bottom line, columns: *light sensitive* and *refraction*).

Remark 2 *Some p-predicate values in the example could be debated. For example, if* refract *is undefined for lens-film distance, one might claim that it should be undefined for aperture as well. We shall return to such variations later, in section 8, in the context of pullbacks.*

The choice of w-elements, connotations, and the p-predicate is crucial for the perceptions and the analogy of this example. In other contexts either the whole eyeball would make a single w-element, or, on the other hand, the sense organs and their accessories could be 'chunked' in a different manner. Other perceptions could connote the same w-elements with chemical properties, attributes of form and design, etc. In these cases it would probably be harder or impossible to define an analogy. (In science, functional resemblances are often more likely to be fundamental than qualitative ones.) Mitchell [50] and French [22] describe computer systems that sort out the connotations and generate mappings that capture possible analogies in given environments, thus modeling the interaction between perceptions and connotations that underlies the creation of analogies.

Example consequences that could be derived from the analogy between the camera and the eye could be:

- It may explicate the eye environment in terms of the camera environment. A perceptive intelligence that has a model of the functioning of a camera can efficiently deduce the functioning of the eye. For example, it could deduce that the image that is created on the retina should be inverted, because it is inverted on the film. This could give rise to a question 'Do humans see everything inverted ?' which is not an unintelligent question to ask.

- When a perceptive intelligence that analogizes between the camera and the eye interacts with a human expert, this intelligence could perhaps be expected to follow a statement like 'A film is never retina-sensitive'.

- The analogy may be used to suggest principles of functioning for similar optical instruments. This issue will be elaborated in section 8 in the context of pullbacks.

5 Interpretive and Literal-Analogical p-morphisms

In this study we are particularly interested in p-morphisms that capture cognitive transitions between distinct environments that are similar in some *analogical* sense, like the example of section 4 concerning an analogy between the camera and the eye. The basic categorical tools for that transition are p-morphisms. P-morphisms can be classified, among others, by subcategories of $\mathcal{P}rc$.

Given an environment $\mathcal{E}$, all possible perceptions $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ with that fixed $\mathcal{E}$ form a subcategory $\mathcal{P}rc_{\mathcal{E}}$. (There is actually a family of such subcategories, one for every possible $\mathcal{E}$.) $\mathcal{P}rc_{\mathcal{E}}$ is the category that was studied in [5, 4]. P-morphisms in $\mathcal{P}rc_{\mathcal{E}}$ are such that h is the identity on $\mathcal{E}$ (see definition 2), these are *interpretive* cognitive transitions between perceptions. In interpretive p-morphisms domain connotations are transferred to codomain connotations that represent another perception of the same environmental phenomena.

There is a subcategory which is 'dual' to $\mathcal{P}rc_{\mathcal{E}}$: The generic subcategory, $\mathcal{P}rc_{\mathcal{I}}$, that is introduced now. It is constructed by fixing some $\mathcal{I}$, a set of connotations, and considering all possible perceptions $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ with that $\mathcal{I}$. P-morphisms here are those where (in terms of definition 2) h is the identity on $\mathcal{I}$. They capture transitions between different environments *while fixing the internal apparatus*. We shall call these *literal-analogical* p-morphisms, *l-analogies* for short. In l-analogies domain w-elements are transferred to codomain w-elements with the same connotations. The analogy between the camera and the eye is formalized by an l-analogy. In [3] creative design processes are also formalized in $\mathcal{P}rc_{\mathcal{I}}$, where w-elements in the domain perception provide examples, similes, and raw material for the designed w-elements in the codomain perception.

There are other possible cognitive transitions that are captured by p-morphisms in $\mathcal{P}rc_{\mathcal{I}}$ that are not analogies in a narrow sense, because they are transitions between w-elements that are, essentially, the same. (As Magnan and Reyes have observed [48], categorical constructs are able to provide means to avoid overdeterminations. In our case, a p-morphism can be regarded as a generalized notion of an analogy.) Consider a book store environment, where w-elements are books, magazines, and newspapers. A 'browsing' agent, with perception $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$, needs to know whether a given w-element, the magazine w, is, say, the *December issue*. In terms of definition 1, $\varrho(w, \textit{December\_issue}) = u$, and this perception has to be improved to the point where that value is definite (t or f). There are three typical methods to go about this:

- A pro-active store agent may perhaps take the initiative to put big red stickers that say *December Issue*. In that case the environment undergoes a transition. It is an l–analogy, $h : \mathcal{P} \to \langle \mathcal{E}_1, \mathcal{I}, \varrho_1 \rangle$, that captures a manipulation of the environment that replaces the magazine w by $h(w)$ that has the sticker. This transition is cognitively conceived to improve browsers' perceptions. It does not change $\mathcal{P}$'s set of connotations, however, it is more likely that $\varrho_1(h(w), \textit{December\_issue}) = t$.

- Depending on its communicative capabilities, the agent can perhaps request the assistance of a store agent. Possessing a different set of connotations, and, hence, a different perception of the same store environment, the store agent gives a definite answer that is based on its own perception of this environment, $\mathcal{P}_0 = \langle \mathcal{E}, \mathcal{I}_0, \varrho_0 \rangle$. As an example, $\mathcal{P}_0$ may have $\varrho_0(w, \textit{from\_top\_shelf}) = t$, in that case one gets an interpretive p-morphism $h : \mathcal{P} \to \mathcal{P}_0$ where $h(\textit{December\_issue}) = \textit{from\_top\_shelf}$. By the no–blur structure preservation condition on p-morphisms, whenever w is the December issue for $\mathcal{P}$, w is from the top shelf for $\mathcal{P}_0$, and whenever w is not the December issue for $\mathcal{P}$, w is not from the top shelf for $\mathcal{P}_0$. In this case the environment is not changed, but there is an interpretation based on a transition to another set of connotations.

- The agent may peek at the dates on the cover and determine a definite (t or f) value for $\varrho(w, \textit{December\_issue})$. In this case both the environment and the connotations are not changed, only the perception predicate undergoes a transition, which may perhaps be enabled by perceptual sensory-motor-neural capabilities,

The last item is captured by a *pure improvement* as in definition 4 below: a p-morphism that is both in $\mathcal{P}rc_{\mathcal{E}}$ and in $\mathcal{P}rc_{\mathcal{I}}$. There is neither an interpretation, nor an analogy. It captures an unblurring of perception within the same environment and representation.

Definition 4 *A p-morphism* $h : \mathcal{P}_1 \to \mathcal{P}_2$ *is a pure* improvement *if* h *is the identity both on* $\mathcal{E}$ *and on* $\mathcal{I}$*, but* $\varrho_1 \neq \varrho\_2$.

By the no–blur condition on p-morphisms, perception is more defined, namely *improved*, in $\mathcal{P}_2$.

The three example transitions (i.e. p-morphisms) above have $\mathcal{P}$ as domain. They are three different methods that could perhaps improve the browsing agent's perception. All are formalized within the same basic simple formalism, and can be initiated by perceptive intelligent artifacts, depending on their perceptual sensory-motor-neural capabilities and their cognitive and communicative skills.

Similar to interpretations in $\mathcal{P}rc_{\mathcal{E}}$, the mathematical properties of l-analogies in $\mathcal{P}rc_{\mathcal{I}}$, (such as being an isomorphism, one-to-one, many-to-one, onto, impossible etc.), carry meticulous information about how close the environments are, and pinpoints differences between them. For example, transitions into larger environments could be captured by l-analogies that are not onto. In other cases, different perceptions could break the same environment into different w-elements, and hence cognitive transitions to larger environmental chunks could sometimes be captured by many-to-one l-analogies. Where one perception perceives three distinct books, another perception might perceive one trilogy. Likewise, many volumes could constitute a single encyclopedia, or a single (arbitrarily large) topic-section of books. Some interesting examples of manipulations of environments and representations to improve perception are given by Hutchins [34].

6 Metaphors as Factorizations of Analogies

By (the extended) definition 2 of p-morphisms, a general p-morphism is both interpretive and analogical: domain w-elements are mapped to *analogical* codomain w-elements, with codomain connotations that are *interpretations* of the domain connotations.

Example 1 *Consider an analogy between a simplistic perception of light and a simplistic perception of sound. For the light perception the environment consists of w-elements that are perceived by the light that they emit, while for the sound perception w-elements are perceived by their sound. The light perception connotes its w-elements by either one of {infrared, visible, ultraviolet}, while the sound perception connotes its w-elements by either one of {infrasonic, audible, ultrasonic}. The analogy h maps: h(infrared)=infrasonic, h(visible)=audible, and h(ultraviolet)=ultrasonic. In this analogy neither the mapping of w-elements, nor the mapping of connotations is the identity. It is not a literal analogy, since there is also an interpretive transfer of connotations involved.*

Example 2 *Analogies that involve both transference of w-elements and transference of connotations could be defined within the same environment and the same connotation set, if* $h : \mathcal{E} \to \mathcal{E}$ *and* $h : \mathcal{I} \to \mathcal{I}$ *are not identities. Linguistic paraphrases, and also the analogies made by the computational models of Mitchell [50] and French [22], seem to call for such a formalization with* p-endomorphisms.

6.1 Metaphors as Factorizations of Analogies: Technicalities

Technically, a factorization of every p-morphism into an interpretation and an l-analogy provides a basic categorical view of the transition.

Definition 5 *Let* $\mathcal{P}_1 = \langle \mathcal{E}_1, \mathcal{I}_1, \varrho_1 \rangle$ *and* $\mathcal{P}_2 = \langle \mathcal{E}_1, \mathcal{I}_2, \varrho_2 \rangle$ *be two perceptions, and let* $h : \mathcal{P}_1 \to \mathcal{P}_2$ *be a p-morphism between them. An* ($\mathcal{E}$-$\mathcal{I}$) *(or* ($\mathcal{I}$-$\mathcal{E}$)*) factorization of h consists of two p-morphisms* $h_\mathcal{E}, h_\mathcal{I}$ *such that:*

1. $h_\mathcal{E} \in \mathcal{P}rc_\mathcal{E}$ *and* $h_\mathcal{I} \in \mathcal{P}rc_\mathcal{I}$.
2. *Either* $h = h_\mathcal{E} \circ h_\mathcal{I}$ *(in that case it is an* ($\mathcal{E}$-$\mathcal{I}$) *factorization),*
   or $h = h_\mathcal{I} \circ h_\mathcal{E}$ *(in that case it is an* ($\mathcal{I}$-$\mathcal{E}$) *factorization).*
3. *The* ($\mathcal{E}$-$\mathcal{I}$) *factorization has the property that if* $h = g' \circ g''$ *is another factorization that satisfies the first two requirements, then there exists a p-morphism* $\psi \in \mathcal{P}rc_\mathcal{E}$ *with* $h_\mathcal{E} \circ \psi = g'$ *and* $\psi \circ g'' = h_\mathcal{I}$. *See figure 1.*
   Dually, the ($\mathcal{I}$-$\mathcal{E}$) *factorization has the property that if* $h = f' \circ f''$ *is another factorization that satisfies the first two requirements, then there exists a p-morphism* $\varphi \in \mathcal{P}rc_\mathcal{I}$ *with* $h_\mathcal{I} \circ \varphi = f'$ *and* $\varphi \circ f'' = h_\mathcal{E}$. *See figure 1.*

Theorem 2 *Let h be a p-morphism as in definition 5.*

1. *An* ($\mathcal{I}$-$\mathcal{E}$) *factorization of h is defined by: (See upper part of figure 1)*

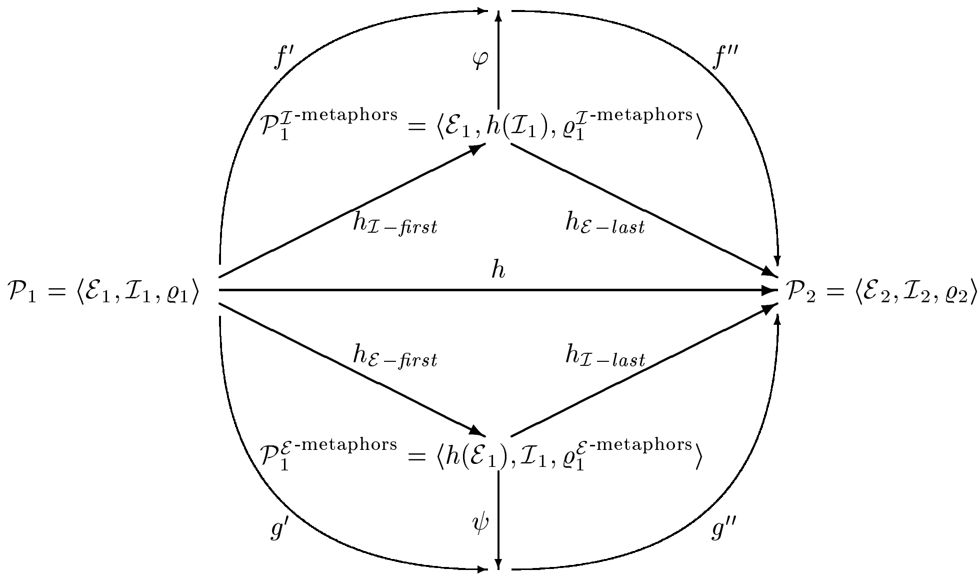

Figure 1: Two Factorizations of a p-morphism

- $h_{\mathcal{I}-first} : \mathcal{P}_1 \to \mathcal{P}_1^{\mathcal{I}\text{-metaphors}} \quad , \quad (\mathrm{w}, \alpha) \mapsto (\mathrm{w}, h(\alpha))$
- $h_{\mathcal{E}-last} : \mathcal{P}_1^{\mathcal{I}\text{-metaphors}} \to \mathcal{P}_2 \quad , \quad (\mathrm{w}, h(\alpha)) \mapsto (h(\mathrm{w}), h(\alpha))$
- $\mathcal{P}_1^{\mathcal{I}\text{-metaphors}} = \langle \mathcal{E}_1, h(\mathcal{I}_1), \varrho_1^{\mathcal{I}\text{-metaphors}} \rangle$
- *Definition of* $\varrho_1^{\mathcal{I}\text{-metaphors}}$*: for all* $\mathrm{w} \in \mathcal{E}_1$*, and for all* $\alpha \in \mathcal{I}_1$*:*

$$\varrho_1^{\mathcal{I}\text{-metaphors}}(\mathrm{w}, h(\alpha)) = \begin{array}{l} t \text{ if } \exists \beta \in \mathcal{I} \text{ such that } h(\beta) = h(\alpha) \text{ and } \varrho_1(\mathrm{w}, \beta) = t \\ f \text{ if } \exists \beta \in \mathcal{I} \text{ such that } h(\beta) = h(\alpha) \text{ and } \varrho_1(\mathrm{w}, \beta) = f \\ u \text{ otherwise} \end{array}$$

2. *Dually, an* ($\mathcal{E}$-$\mathcal{I}$) *factorization of* h *is defined by: (See lower part of figure 1)*

- $h_{\mathcal{E}-first} : \mathcal{P}_1 \to \mathcal{P}_1^{\mathcal{E}\text{-metaphors}} \quad , \quad (\mathrm{w}, \alpha) \mapsto (h(\mathrm{w}), \alpha)$
- $h_{\mathcal{I}-last} : \mathcal{P}_1^{\mathcal{E}\text{-metaphors}} \to \mathcal{P}_2 \quad , \quad (h(\mathrm{w}), \alpha) \mapsto (h(\mathrm{w}), h(\alpha))$
- $\mathcal{P}_1^{\mathcal{E}\text{-metaphors}} = \langle h(\mathcal{E}_1), \mathcal{I}_1, \varrho_1^{\mathcal{E}\text{-metaphors}} \rangle$
- *Definition of* $\varrho_1^{\mathcal{E}\text{-metaphors}}$*: for all* $\mathrm{w} \in \mathcal{E}_1$*, and for all* $\alpha \in \mathcal{I}_1$*:*

$$\varrho_1^{\mathcal{E}\text{-metaphors}}(h(\mathrm{w}), \alpha) = \begin{array}{l} t \text{ if } \exists x \in \mathcal{E} \text{ such that } h(x) = h(\mathrm{w}) \text{ and } \varrho_1(x, \alpha) = t \\ f \text{ if } \exists x \in \mathcal{E} \text{ such that } h(x) = h(\mathrm{w}) \text{ and } \varrho_1(x, \alpha) = f \\ u \text{ otherwise} \end{array}$$

Proof. First note that, by definition, the factors are legitimate p-morphisms, because h is. We show that the properties required by definition 5 do hold for the ($\mathcal{E}$-$\mathcal{I}$) factorization of h, as illustrated in the lower part of figure 1:

1. By definition, $h_{\mathcal{E}-first} \in \mathcal{P}rc_{\mathcal{I}}$ and $h_{\mathcal{I}-last} \in \mathcal{P}rc_{\mathcal{E}}$.

2. By definition, $h = h_{\mathcal{E}-first} \circ h_{\mathcal{I}-last}$.

3. Let $h = g' \circ g''$ be another ($\mathcal{E}$-$\mathcal{I}$) factorization of h. Define $\psi \in \mathcal{P}rc_{\mathcal{E}}$ by $h(\alpha) \mapsto g'(\alpha)$. The mapping is legitimate since: $h(\alpha) = h(\beta) \implies g' \circ g''(\alpha) = g' \circ g''(\beta)$. Since $g'' \in \mathcal{P}rc_{\mathcal{I}}$, then also $g'(\alpha) = g'(\beta)$. ψ is no-blur by 'the minimality' of the definition of $\varrho_1^{\mathcal{E}\text{-metaphors}}$, and by g' being a p-morphism. It is also easy to see that $h_{\mathcal{E}-first} \circ \psi = g'$ and $\psi \circ g'' = h_{\mathcal{I}-last}$.

Proof of the required properties for the ($\mathcal{I}$-$\mathcal{E}$) factorization of h, as illustrated in the upper part of figure 1, is dual. □

Corollary 1 *Each one of the ($\mathcal{E}$-$\mathcal{I}$) and ($\mathcal{I}$-$\mathcal{E}$) factorizations of theorem 2 is unique up to isomorphism.*

Proof. If there are two such factorizations then ψ (or φ) of item 3 in definition 5 is a reversible p-morphism, and hence a p-isomorphism. □
The intuitive idea is to gain access to the intermediate, *Metaphorical Perceptions*:

- $\mathcal{P}_1^{\mathcal{I}\text{-metaphors}} = \langle \mathcal{E}_1, h(\mathcal{I}_1), \varrho_1^{\mathcal{I}\text{-metaphors}} \rangle$ consists of $\mathcal{P}_1$ w-elements with *metaphorical connotations* that are inspired by $\mathcal{P}_2$.
- $\mathcal{P}_1^{\mathcal{E}\text{-metaphors}} = \langle h(\mathcal{E}_1), \mathcal{I}_1, \varrho_1^{\mathcal{E}\text{-metaphors}} \rangle$ consists of *metaphorical w-elements* that are inspired by $\mathcal{P}_2$, with $\mathcal{P}_1$ connotations.

Example 3 *An ($\mathcal{I}$-$\mathcal{E}$) factorization of h of the camera-eye analogy from section 4 yields a perception of a camera with metaphorical w-elements that are 'borrowed' from the eye environment. One relates to a camera environment but perceives a* pupil *instead of an* aperture, *a* cornea *instead of a* camera lens, *etc. When asked whether the* eye lens *of the camera refracts, the answer is undefined. This follows from the 'minimal' definition of the metaphorical p-predicates in theorem 2, and from the p-predicate of $\mathcal{P}_{\text{Cam}}$ (see table 1, bottom line, refract column, left side).*

Example 4 *Metaphorical terms such as 'white noise' are based on extensions of perceptions and analogies that are similar in spirit to example 1, where an analogy was proposed, roughly, between visible light and audible sound, and the metaphorical perception of a 'white noise' can be captured by a factorization as in theorem 2.*

Example 5 *When linguistic translations are conceived as analogies, as proposed in section 1, then borrowing words or phrases from another language is based on a metaphorical transition as above.*

6.2 Metaphors as Factorizations of Analogies: Methodological Fallout

From the category theoretical point of view, the factorization is straight forward, and the formulation of definition 5 is standard (item 3 is typically category theoretical). We wanted to state that metaphorical perceptions are based on a blend of constituents from the two sides of an analogy, yet that the interfusion should be circumscribed by the perceptions involved. The categorical tools allowed a precise and testable extension of these intuitions, that can be modeled by an artificial system more readily than the verbal description. The *metaphorical p-predicates* of theorem 2 are 'minimally unblurred' with respect to the domain perception. Technically, this warrants compliance with item 3 of definition 5 (and also the uniqueness of the factorizations). Intuitively, item 3 of definition 5 is a systematization that captures the circumscription of metaphors: they cannot become arbitrarily 'far-fetched'. A specific example of the way that this limitation works was shown above for the camera-eye analogy: The metaphorical perception of the eye lens of the camera leaves undefined the issue of whether it refracts. The metaphor is not carried beyond certain bounds, and these bounds are rigorously defined by the formalism. This is one more instance where the category of artificial perceptions provides us with a reward for the technical efforts.

The corresponding *metaphorical transitions* are formalized by the respective p-morphisms, $h_{\mathcal{I}-first}$ and $h_{\mathcal{E}-first}$. Categorical properties of these p-morphisms can be applied to evaluate the metaphorical transition: Is it one-to-one? Is it rigid (definition 3)? A positive answer to both last questions means, for example, 'simpler' metaphors. If a p-morphism h is already in $\mathcal{P}rc_{\mathcal{E}}$ (or in $\mathcal{P}rc_{\mathcal{I}}$, as in the camera-eye analogy of section 4), then the relevant factor is the identity. This could perhaps be allusive of Nietzsche's observation that literal truth is merely dead or fossilized metaphor.

The proposed factorization is relevant to other perceptual–cognitive issues. One of them is the evasive boundary between literal meaning on one hand, and methaphorical or metonymic meaning on the other hand. Philosophers as well as linguists vary in their views. Lakoff, for example, holds the position that many of our conventional everyday concepts are metaphorical. Indurkhya, on the other hand, holds the position that to qualify as a metaphor, a concept needs to be given an *unconventional* interpretation. The proposed factorization may be applied to bridge the gap between the views: An agent may start out with a perception such as $\mathcal{P}_1$ of theorem 2. Applying an analogical transition h into another perception, like $\mathcal{P}_2$ of theorem 2, and then factorizing that analogy, the agent may perceive with the implied metaphors. If the agent *internalizes* the intermediate metaphorical perception, say $\mathcal{P}_1^{\mathcal{E}\text{-metaphors}} = \langle h(\mathcal{E}_1), \mathcal{I}_1, \varrho_1^{\mathcal{E}\text{-metaphors}} \rangle$ from figure 1, and gets used to employing it on a regular basis, then the metaphors become 'conventional' or even 'literal'. By *internalization* it is meant that in subsequent recognition or usage of the metaphor the agent does not go through the entire process of transition and factorization all over again, but rather that the metaphorical perception becomes a basic perception, with the same status that $\mathcal{P}_1$ had had at the outset. Metaphors may start out as 'unconventional' and become 'conventional'. The degree of conventionality is 'in the eye of the perceiver', depending on how much computational effort (i.e. mappings, p-morphisms, factorizations) is invested in finally identifying one of the three values $\{t, f, u\}$ with instances of $\varrho_1^{\mathcal{E}\text{-metaphors}}(w, \alpha)$. Perceptual states, either individual or shared by a society of agents, undergo continuous dynamic transitions, so that both 'conventionality' and 'unconventionality' are subjective and fluid.

Another perceptual–cognitive problem that could perhaps benefit from the proposed factorization of p-morphisms is the long recognized fact that many concepts (or *words*) can take on an infinite number of different senses, because there is no limit on the number of contexts. Clark [15] argued that context contributes to the meaning of a concept, on each particular occasion of use, in a deep way. Others have made related points. The sense variant of the concept is constructed at the time of perception from some core meaning, in combination with the context in which it occurs. All agree that, although meanings change from agent to agent, and they also change within agents over time and experiences, some critical invariable aspect of meaning is held by individual perceptions and must be shared in order for perceptual transitions to make sense. One issue that remains open for rigorous formalization is how to fix a core invariant meaning across environments, individual representations, and instances of perceptual transitions. The structure preservation condition on p-morphisms binds different representations through environments, and analogical environments through representations. The factorization may untangle the knot. Loosely, if one fixes the interpretation (i.e. $h_{\mathcal{I}}$ is the identity), then one can perform a meaningful analogical transition between environments. Dually, when fixing the environment (i.e. $h_{\mathcal{E}}$ is the identity), one can perform a meaningful interpretive transition between representations. Complex transitions can be 'broken' (i.e. factorized) into (recomposable) smaller steps of this kind, thus, perhaps, grasping the evasive invariable aspect of meaning with a loose and flexible, yet durable, harness.

7 Structural Alignment: Analogizing with Structured Arrows

At the roots of powerful analogies and metaphors lie more than just 'pointwise' mappings. Research background about the significance of *structured* analogies was mentioned in section 2 from several perspectives. Thagard [55, p.81] argues that *... Two analogs are similar to each other at a superficial level if they involve similar concepts... However, powerful analogies involve not just superficial similarities, but also deeper structural relations.*

The proposed categorical formalism will be refined now to capture structural features of analogies. Perceptible structure consists essentially of observed relations among the constituents of a perception. They play an important role in the conception of analogies, and imply restrictions on the manner in which things can be mapped by an analogy. Based on a given perception $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$, the formalism provides tools for capturing observation of lawlike patterns among w-elements of $\mathcal{E}$ as well as among connotations of $\mathcal{I}$. We start by introducing a quasi ordering on w-elements. Definitions 6 and 7 below are novel in the proposed categorical setting.

Definition 6 *Let $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ be a perception. A w-element $x \in \mathcal{E}$ is* subjacent *to another w-element $y \in \mathcal{E}$, designated $x \trianglelefteq y$, if, for all connotations α in $\mathcal{I}$, $\varrho(x, \alpha) = t$ implies that $\varrho(y, \alpha) = t$ and $\varrho(y, \alpha) = f$ implies that $\varrho(x, \alpha) = f$.*

Definition 7 *Let $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ be a perception. Two w-elements x, y in $\mathcal{E}$ are* congeneric *if, for all connotations α in $\mathcal{I}$, $\varrho(x, \alpha) = \varrho(y, \alpha)$.*

It is easy to see that congenerics are subjacent one to the other, and that the congeneric relation is an equivalence relation. The subjacency relation is a quasi ordering on $\mathcal{E}$, and a partial ordering on the equivalence classes of the congeneric relation. Example Subjacencies of w-elements in $\mathcal{P}_{\text{Cam}}$ are, for instance, aperture $\trianglelefteq$ shutter, and also shutter $\trianglelefteq$ outside actions. Example Subjacencies of w-elements in $\mathcal{P}_{\text{Eye}}$ are, for instance, pupil $\trianglelefteq$ iris, and also iris $\trianglelefteq$ brain stem reflexes. These patterns are, indeed, meaningful for understanding the structure and the functioning of the camera and the eye.

Parallel relations between connotations were already introduced and discussed in [4, p.203-207]. They are repeated below for easy reference.

Definition 8 *Let $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ be a perception. A connotation $\alpha \in \mathcal{I}$ subsumes another connotation $\beta \in \mathcal{I}$, designated $\alpha \trianglelefteq \beta$, if, for all w-elements w in $\mathcal{E}$, $\varrho(\mathrm{w}, \alpha) = t$ implies that $\varrho(\mathrm{w}, \beta) = t$ and $\varrho(\mathrm{w}, \beta) = f$ implies that $\varrho(\mathrm{w}, \alpha) = f$.*

Definition 9 *Let $\mathcal{P} = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$ be a perception. Two connotations α, β in $\mathcal{I}$ are* synonyms *if, for all w-elements w in $\mathcal{E}$, $\varrho(\mathrm{w}, \alpha) = \varrho(\mathrm{w}, \beta)$.*

It is easy to see that synonyms subsume one another, and that synonymity is an equivalence relation. The subsumption relation is a quasi ordering on $\mathcal{I}$, and a partial ordering on the equivalence classes of synonyms. Perceptions that are *unique up to synonyms* were studied in [5, p.291-295]. In a dual manner environments that are *unique up to congenerics* can be defined, mutatis mutandis. The intuitive idea is that more duplicates of the same constituent do not feature a significant difference in the structural aspect of a perception.

In the perceptions $\mathcal{P}_{\text{Cam}}$ and $\mathcal{P}_{\text{Eye}}$ from table 1, one can easily observe several subsumptions of connotations. Subsumptions that hold in both perceptions are, for example, *control of another w-element* $\trianglelefteq$ *dynamic change*, *light amount regulation* $\trianglelefteq$ *dynamic change*, *focus adjustment* $\trianglelefteq$ *dynamic change*. These patterns are, indeed, meaningful for understanding the functioning of the camera and the eye.

Hence, introduction of a structural element both into the internal representation $\mathcal{I}$ on one hand, and into the environment $\mathcal{E}$ on the other hand, occurs when a perceptive intelligence observes lawlike patterns between the constituents of its perception. (The definitions are inspired by the definitions that Lukasiewicz gave to the biconditional and to the conditional in his 3-valued logic [46, 45]. The choice is discussed in [4, p.192-194].)

The subjacency and subsumption relationships do not necessarily represent *causal* relationships, or *isa* relationships, although these are particular possibilities. Observation of a lawlike pattern is independent of an explanation pertaining to *why* or *how* the pattern holds [6]. The observation may, indeed, stimulate reasoning higher-level processes to track the causes or the nature of the pattern, but this is a separate cognitive process. Understanding, for example, how or why the iris is subjacent to brain stem reflexes is a complicated issue that may be studied separately.

Intelligence is typically marked by a discerning perception and understanding of its environment, and analogy making ought to benefit from these capabilities and demonstrate them. When a perceptive intelligence observes lawlike patterns

[6]Perception follows, in this context, the scientific enquiry principle *Hypotheses non fingo* [11, Newton, page 261].

in its environment, it would be natural to incorporate the observations into its cognitive processes. The correlate in a category theoretical setting is to preserve the structure when applying a categorical construct. The definition of *monotonicity* from [4, p.206] is extended now to include also the structure that was defined on the environment $\mathcal{E}$ (definitions 6 and 7).

Definition 10 *Let* $\mathcal{P}_1 = \langle \mathcal{E}_1, \mathcal{I}_1, \varrho_1 \rangle$ *and* $\mathcal{P}_2 = \langle \mathcal{E}_1, \mathcal{I}_2, \varrho_2 \rangle$ *be two perceptions. A p-morphism* $h : \mathcal{P}_1 \to \mathcal{P}_2$ *is* Monotone *if, for all* α, β *in* $\mathcal{I}_1$, $\alpha \trianglelefteq \beta$ *implies that* $h(\alpha) \trianglelefteq h(\beta)$, *and, for all* x, y *in* $\mathcal{E}_1$, $x \trianglelefteq y$ *implies that* $h(x) \trianglelefteq h(y)$.

In the *Monotone subcategory of perceptions* arrows are restricted to monotone ones, and hence all categorical constructs preserve the involved structure. Rigid p-morphisms as in definition 3 are always monotone. Non-monotonicity occurs only (but not necessarily) when there is some unblurring of perception. When $x \trianglelefteq y$, and for some $\alpha \in \mathcal{I}$ it so happens that $\varrho(x, \alpha) = \varrho(y, \alpha) = u$, then if $h(x)$ has the connotation $h(\alpha)$ but $h(y)$ does not, then this is a typical case of non-monotonicity, since $h(x)$ is not subjacent to $h(y)$. In the case where an analogy is described by a monotone p-morphism, this analogy is systematized by a mapping that preserves structure, and so are the metaphors that are implied by the factorizations of a monotone analogy.

In the l-analogy between the camera and the eye, some subsumptions of connotations and some subjacencies of w-elements are preserved. Among them are the examples that were provided above following the definitions. The example subsumptions of connotations are literally preserved: *control of another w-element* $\trianglelefteq$ *dynamic change*, *light amount regulation* $\trianglelefteq$ *dynamic change*, *focus adjustment* $\trianglelefteq$ *dynamic change*. The example subjacencies of w-elements are also preserved by the analogy: aperture $\trianglelefteq$ shutter and also pupil $\trianglelefteq$ iris, shutter $\trianglelefteq$ outside actions and also iris $\trianglelefteq$ brain stem reflexes. This structural alignment actually means that the implied structure and functioning of the camera and the eye are similar, making the analogy between the camera and the eye a meaningful analogy.

The bad news is that the p-morphism that describes the l-analogy between the camera and the eye is, however, *not* monotone. As mentioned above, non-monotonicity may occur only when there is some unblurring of perception. Hence the row of table 1 that should be examined carefully is the bottom row, namely subjacencies that involve either the lens-film distance or the eye lens+muscle, and subsumptions that involve either *light sensitivity* or *refraction*. There is, indeed, non-monotonicity there: lens-film distance $\trianglelefteq$ outside actions but eye lens $\ntrianglelefteq$ brain stem reflexes because ϱ(eye lens,*refraction*)$=t$, but ϱ(brain stem reflexes,*refraction*)$=f$. It follows that one has to omit the *refraction* connotation or accept an analogy that is not monotone.

The principal difference between a camera and an eye has to do with the process of focus adjustment, and the proposed formal setting shows sensitivity to this difference, providing additional support that it is an effective formalism. The mapping of lens-film distance to the flexible eye lens is not rigid (definition 3). In particular, the l-analogy is not a p-isomorphism.

What happened? non-rigidity, and the resulting non-monotonicity, occurred exactly at the point where the analogy is not perfect. When an analogy is captured by a p-morphism, then the monotonicity of that p-morphism happens to be a touchstone for the 'uniformity' of the analogy, and the pigeon-hole of non-monotonicity provides meticulous information about the point where the

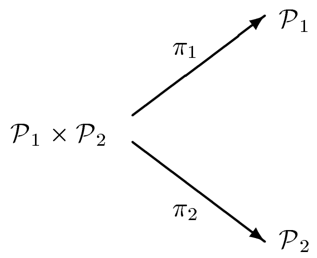

Figure 2: Start by Matching Everything with Everything: P-Product

analogy 'slips' [7]. P-morphisms, monotonicity and rigidity are precise, applicable and testable tools of analysis that are enabled by the categorical formalism.

8 Analogizing with Pullbacks

The analogy between the camera and the eye still 'feels' to be a good one. Let us see if the categorical tools are capable of systematizing the following argument: '*In both systems there is focus adjustment, which may be performed in different ways*'. What one actually claims in this argument is that if one avoids over determination, by not specifying exactly how focus is adjusted, then at that level of perception, the camera and the eye are analogous.

The cognitive process that underlies this argument consists of first generalizing the two perceptions into a third, abstracted, perception of a generalized 'image generating system'. Such a system consists of abstracted w-elements that should be able to produce an image on a light sensitive medium while performing refraction, adjusting focus, regulating light, etc. The camera and the eye become particular substitution instances of this general schema. The generalized perception has an 'abstract environment' that corresponds to what Glynn and al. [26] entitle a *superordinate* system, Thagard [55] entitles *analogical schema*, and Gentner entitles the generation of that generalized perception as the *subprocess of abstraction*. The loose verbal description can be captured by well developed tools of category theory. This will be done now.

8.1 Schema abstraction with P-Products and P-Pullbacks

The definitions of *p-product* and *p-pullback* was introduced in [5, p.288-296] for the subcategory $\mathcal{P}rc_{\mathcal{E}}$. The definitions are now extended to the entire category $\mathcal{P}rc$. A product of two perceptions is their 'minimal change common blur'. Loosely, each perception is blurred exactly to the point where there is no conflict with the other perception.

Theorem 3 *The product of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the perception*

$$\mathcal{P}_1 \times \mathcal{P}_2 = \langle \mathcal{E}_1 \times \mathcal{E}_2, \mathcal{I}_1 \times \mathcal{I}_2, \varrho_1 \times \varrho_2 \rangle$$

Where the set of w-elements is the set product of the sets of w-elements, the set of connotations is the set product of the sets of connotations, and the p-predicate

[7] The term 'slippage' was coined by Hofstadter and his colleagues [31, 50, 22]

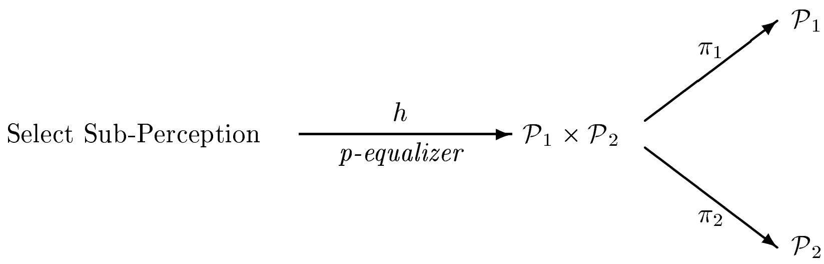

Figure 3: Selecting Analogs

$\varrho_1 \times \varrho_2$ is defined as follows:

$$\varrho_1 \times \varrho_2((\mathrm{w}_1, \mathrm{w}_2), (\alpha_1, \alpha_2)) = \begin{cases} t & \text{if and only if } \forall \imath = 1,2 \;\; \varrho_\imath(\mathrm{w}_\imath, \alpha_\imath) = t \\ f & \text{if and only if } \forall \imath = 1,2 \;\; \varrho_\imath(\mathrm{w}_\imath, \alpha_\imath) = f \\ u & \text{otherwise} \end{cases}$$

The associated projections are (see figure 2):
$\pi_\imath : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathcal{P}_\imath$, *where* $\pi_\imath(\alpha_1, \alpha_2) = \alpha_\imath$ *and* $\pi_\imath(\mathrm{w}_1, \mathrm{w}_2) = \mathrm{w}_\imath$.

The proof that this is indeed a categorical product, and that the p-product always exists, is essentially the same as the proof in [5, p.289-290]. In the subcategory $\mathcal{P}rc_\mathcal{E}$ all p-morphisms identity-map $\mathcal{E}$ to itself, so that the product environment $\mathcal{E} \times \mathcal{E}$ may be degenerated into $\mathcal{E}$.

In the general case, most of the p-predicate values in a p-product are going to be undefined, since most pairs should consist of two essentially different coordinates. For example, if one constructs the p-product of the perceptions of the camera and the eye: $\mathcal{P}_{\text{Cam}} \times \mathcal{P}_{\text{Eye}}$, then a juxtaposition of, say, outside actions and retina, is meaningless. There is no similarity between them. This total absence of similarity is captured formally by the undefined p-predicate value for all pairs of connotations (α_1, α_2): $\varrho_1 \times \varrho_2((\text{outside actions}, \text{retina}), (\alpha_1, \alpha_2)) = u$. The undefined value can be easily computed from the values in table 1 and from the definition of $\varrho_1 \times \varrho_2$ in theorem 3.

The intuitive idea behind p-products is to:

1. First try matching (i.e. analogizing) 'everything with everything' by juxtaposition of all possible pairs in the cartesian products $\mathcal{E}_1 \times \mathcal{E}_2$ and $\mathcal{I}_1 \times \mathcal{I}_2$.

2. The next step will be to 'sort out' only the pairs that feature perceptible connaturalness, *unlike* the pair (outside actions,retina), and more like the pairs that share rows in table 1.

The p-product construction systematizes the first step above. A *p-pullback* will systematize the second step (see figure 3). Before going into the rigorous formalism, an informal description might help: Pointing out a subset of w-elements and a subset of connotations in a perception (e.g. in the p-product) can be simply regarded as an inclusion p-morphism: one-to-one and rigid. This happens to be the definition of a p-equalizer as defined for $\mathcal{P}rc_\mathcal{E}$ in [5, p.285]. An extension to the entire category $\mathcal{P}rc$ is straight forward: A p-equalizer is a one-to-one and rigid p-morphism. The selection provides a new perception that offers possible analogs because it consists of selected pairs from the p-products that feature similarity. (It still remains to define what one means exactly by 'pairs that feature similarity'.) The selection process is systematized by the p-equalizer h in the diagram of figure 3.

Definition 11 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two perceptions. A* superordinate *perception that they generate* [8]*, $\mathcal{P}_1 \sqcap \mathcal{P}_2 \subseteq \mathcal{P}_1 \times \mathcal{P}_2$, consists of w-elements $(\mathrm{w}_{k_1}, \mathrm{w}_{k_2})$ and connotations $(\alpha_{j_1}, \alpha_{j_2})$ from the product, that feature perceptible connaturalness as follows:*

1. *For all connotation pairs $(\alpha_{j_1}, \alpha_{j_2})$, and for all w-element pairs $(\mathrm{w}_{k_1}, \mathrm{w}_{k_2})$, either $\underline{\forall \imath = 1,2\ \varrho_\imath(\mathrm{w}_{k_\imath}, \alpha_{j_\imath}) \neq f}$, or $\underline{\forall \imath = 1,2\ \varrho_\imath(\mathrm{w}_{k_\imath}, \alpha_{j_\imath}) \neq t}$.*

2. *Let $\mathcal{P}_1 \sqcap \mathcal{P}_2 = \langle \mathcal{E}, \mathcal{I}, \varrho \rangle$.*

   - *For every connotation α in this perception there exists at least one w-element w_α in this perception, such that $\varrho(\mathrm{w}_\alpha, \alpha) \neq u$.*
   - *Likewise, For every w-element w in this perception, there exists at least one connotation α_w in this perception, such that $\varrho(\mathrm{w}, \alpha_\mathrm{w}) \neq u$.*

Item 1 warrants that the coordinates of the selected pairs are not dissimilar. Technically, this condition is necessary and sufficient for the p-pullback theorem 4 below. For example, the pair that was mentioned before, (outside actions,retina), cannot be included in the superordinate perception $\mathcal{P}_{\mathrm{Cam}} \sqcap \mathcal{P}_{\mathrm{Eye}}$, because for all the connotations $\alpha \in \{$*regulate light*,*adjust focus*, *dynamic change*,*control of another object*$\}$, $\varrho_{\mathrm{Cam}}(\text{outside actions}, \alpha) = t$ but $\varrho_{\mathrm{Eye}}(\text{retina}, \alpha) = f$, in contradiction with the condition of item 1.

Item 2 requires the existence of a more definite, perceptible, similarity between coordinates. A superordinate perception $\mathcal{P}_{\mathrm{Cam}} \sqcap \mathcal{P}_{\mathrm{Eye}}$, that can be generated by the perceptions of the camera and the eye, may consist of w-elements that are the pairs that share lines in table 1, namely: (outside actions,brain stem reflexes) (shutter,iris), (aperture,pupil), (film,retina), (amera lens,cornea), (lens-film distance,eye lens+muscle). It is easy to verify that they answer both conditions.

By definition of the p-product p-predicate, the condition of item 2 ensures that for every connotation pair $(\alpha_{j_1}, \alpha_{j_2})$ there exists at least one w-element pair $(\mathrm{w}_{j_1}, \mathrm{w}_{j_2})$ such that $\forall \imath = 1,2\ \varrho_\imath(\mathrm{w}_{j_\imath}, \alpha_{j_\imath}) = t$ or $\forall \imath = 1,2\ \varrho_\imath(\mathrm{w}_{j_\imath}, \alpha_{j_\imath}) = f$. Intuitively, in the first part of item 2, the superordinate w-element $w = (\mathrm{w}_{j_1}, \mathrm{w}_{j_2})$ suggests a constituent that underlies the connaturalness of α_{j_1} and α_{j_2}. This constituent may justify compromises that might be introduced by weaker similarities (or, rather, non-dissimilarities) of other constituents that comply only with the first condition. The second part of item 2 is the dual requirement for analogical w-elements. For example, in a superordinate perception $\mathcal{P}_{\mathrm{Cam}} \sqcap \mathcal{P}_{\mathrm{Eye}}$, that might be generated by the perceptions of the camera and the eye, the connotation *adjust focus* is a constituent that underlies a certain connaturalness of lens-film distance and eye lens+muscle. This justifies any compromises that are introduced by weaker similarities (or, rather, non-dissimilarities). They have to do with whether either one of lens-film distance or eye lens+muscle has the connotations *light sensitivity* and *refraction*.

[8] There are probably cases of perceptions that can generate more than one superordinate perception.

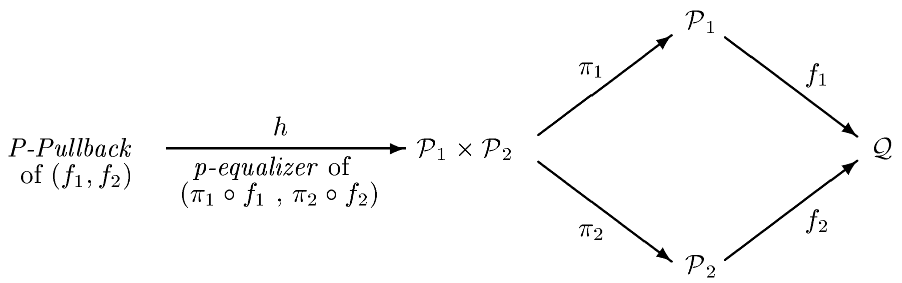

Figure 4: Looking for Analogs: P-Pullback

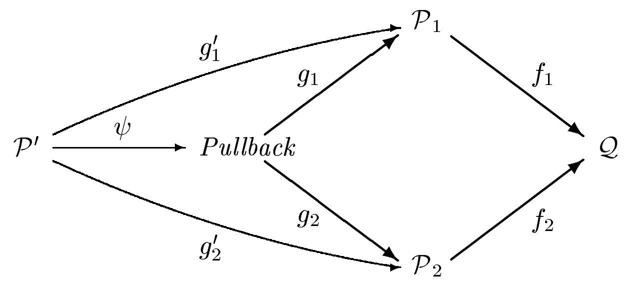

Figure 5: Categorical Pullback

Remark 3 *Whether or not it is enough to analogize with w-elements and connotations that comply only with* negative *definite p-predicate values (f) in item 2 is hard to answer at a global categorical level. This is related to issues that are discussed by Goodman in [27]. It is possible to restrict the definition to the existence of* α_{W} *such that* $\varrho(\mathrm{w}, \alpha_{\mathrm{W}}) = t$*, and to* w_α *such that* $\varrho(\mathrm{w}_\alpha, \alpha) = t$*, because, from the technical point of view, item 1 of the definition is already necessary and sufficient for the pullback construction of theorem 4 below.*

Now, from the category theoretical point of view, a combination of a product and an equalizer yields a pullback, as shown in figure 4. The advantages of the formalization by a pullback follow from the categorical property of pullbacks, which is illustrated in figure 5, and defined as follows:

> Given arrows $f_1 : \mathcal{P}_1 \to \mathcal{Q}$ and $f_2 : \mathcal{P}_2 \to \mathcal{Q}$, the commutative diagram of figure 5, where g_1, g_2 have the same domain, and $g_1 \circ f_1 = g_2 \circ f_2$, is a *pullback*, if it has the property that any other commutative diagram (where $g_1' \circ f_1 = g_2' \circ f_2$) can be completed by a unique ψ.

One import of the pullback property to the process of schema abstraction that is being systematized here, is that the pullback perception is minimally blurred: Connotations of w-elements in the superordinate perception that need not be blurred remain defined. The abstract schema generalizes, but avoids determination only there where this is absolutely necessary (e.g. in the case of whether lens-film distance or eye lens+muscle have the connotations *light sensitivity* and *refraction*) [9].

[9] Readers interested in category theoretical workings are invited to compare the formulation of the pullback property with the formulation of the 'minimality' of metaphors in item 3 of definition 5.

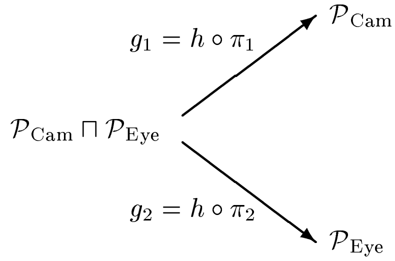

Figure 6: Schema Abstraction for the Camera and the Eye

Theorem 4 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two perceptions. A superordinate perception as in definition 11 can be effected by a categorical pullback construction.*

Proof. For every pair that should be introduced into the superordinate perception, define the p-morphisms $f_\imath$ of the diagram in figure 4 to map the coordinates of the pair to the same element of (a trivially chosen) $\mathcal{Q}$. The first condition of definition 11 is necessary and sufficient for that, so that the $f_\imath$'s should comply with the no-blur condition of definition 2. The superordinate perception is then defined by the p-pullback of f_1, f_2, that is categorically defined by the p-equalizer of $(\pi_1 \circ f_1, \pi_2 \circ f_2)$. It consists of the desired subsets of pairs from the product $\mathcal{P}_1 \times \mathcal{P}_2$. □

A superordinate perception that can be generated by the perceptions of the camera and the eye with connotation pairs as described above consists of w-elements that are the pairs that share lines in table 1. The connotations of the superordinate perception are pairs of identical connotations, so that pairs can be degenerated into the original single connotation. (Actually, this is a construction in the subcategory $\mathcal{P}rc_{\mathcal{I}}$ where $\mathcal{I}$ is fixed, so the construction is based on set products of environments only.)

It is already known from the former section that weak similarities (or, rather, non-dissimilarities) are to be expected from the bottom row of table 1. Indeed, the only cases of weaker similarity are associated with the w-element pair w=(lens-film distance,eye lens+muscle). The following p-predicate values answer only to the weaker condition in item 1 of definition 11: $\varrho_\imath(w_\imath, \textit{light\_sensitive}) \neq t$, and $\varrho_\imath(w_\imath, \textit{refraction}) \neq f$. However, the following p-predicate values answer also to the stronger condition in item 2 of that definition: $\varrho_\imath(w_\imath, \textit{adjust\_focus}) = t$, $\varrho_\imath(w_\imath, \textit{dynamic\_change}) = t$, $\varrho_\imath(w_\imath, \textit{regulate\_light}) = f$, and $\varrho_\imath(w_\imath, \textit{control\_of\_other\_object}) = f$, featuring a definite kind of similarity. There is, hence, a schema abstraction that captures the idea of a w-element that changes dynamically and takes care of focusing (and does not have to do with light regulation and control of other objects).

There are projection l-analogies from this superordinate perception to the camera perception: $g_1 = h \circ \pi_1 : \mathcal{P}_{\text{Cam}} \sqcap \mathcal{P}_{\text{Eye}} \to \mathcal{P}_{\text{Cam}}$, and to the eye perception: $g_2 = h \circ \pi_2 : \mathcal{P}_{\text{Cam}} \sqcap \mathcal{P}_{\text{Eye}} \to \mathcal{P}_{\text{Eye}}$, as shown in figure 6. The p-predicate for (lens-film distance,eye lens+muscle) with *refraction* and with *light sensitivity* unblurs in different ways for each of the projecting l-analogies, providing a formal account of the difference between the camera and the eye.

8.2 Properties of the Superordinate Perception

In section 7 the concept of structural alignment in analogies was systematized within the categorical framework by the monotonicity of the relevant p-morphisms. A natural question that may be asked now is what happens to the structure when a superordinate perception is generated by a p-pullback as above. In particular, we ask whether the projection l-analogies from the superordinate perception to the concrete perceptions (that gave rise to the whole process) are monotone. The answer is positive, as shown in the following:

Theorem 5 *Let $\mathcal{P}_1 \sqcap \mathcal{P}_2$ be a superordinate perception.*

- *If (α, α') and (β, β') are connotations in $\mathcal{P}_1 \sqcap \mathcal{P}_2$ such that $\alpha \trianglelefteq \beta$ in $\mathcal{P}_1$ and $\alpha' \trianglelefteq \beta'$ in $\mathcal{P}_2$, then also $(\alpha, \alpha') \trianglelefteq (\beta, \beta')$ in $\mathcal{P}_1 \sqcap \mathcal{P}_2$.*
- *If (x, x') and (y, y') are w-elements in $\mathcal{P}_1 \sqcap \mathcal{P}_2$ such that $x \trianglelefteq y$ in $\mathcal{P}_1$ and $x' \trianglelefteq y'$ in $\mathcal{P}_2$, then also $(x, x') \trianglelefteq (y, y')$ in $\mathcal{P}_1 \sqcap \mathcal{P}_2$.*

Proof. The theorem is proven using the definition of p-products from theorem 3. For example, assume negatively that for some α in $\mathcal{P}_1 \sqcap \mathcal{P}_2$, (x, x') has the connotation α but it is undefined whether (y, y') has the connotation α (in which case $(x, x') \trianglelefteq (y, y')$ does not hold). In that case, by definition of the product p-predicate, either $x \trianglelefteq y$ does not hold in $\mathcal{P}_1$, or $x' \trianglelefteq y'$ does not hold in $\mathcal{P}_2$, in contradiction to the negative assumption. □
The import of the last theorem is that *the superordinate perception captures not only the similarities between constituents, but also the common structure.* Moreover, even in the presence of a weaker similarity between constituents, lawlike patterns, if they are shared, will not be 'erased'.

Corollary 2 *The projection l-analogies from the superordinate perception (the p-pullback) $g_1 = h \circ \pi_1$ and $g_2 = h \circ \pi_2$ are monotone.*

In the case of the superordinate perception $\mathcal{P}_{\text{Cam}} \sqcap \mathcal{P}_{\text{Eye}}$, for example, the schema abstraction got rid of the problematic structure that could not be aligned by the analogy. The mathematical formalism tells us in a precise and testable manner that the generalization was *exactly* enough for that, neither more, nor less. This is yet another instance where one reaps the fruits of the effort invested in the formalism of the category of artificial perceptions.

Of course, the pullback construction could yield an empty superordinate perception, or a perception with very few constituents. Theorem 5 tells us that the process would 'erase' significant structure if it pertained exclusively either to $\mathcal{P}_1$ or to $\mathcal{P}_2$. This would essentially mean that the analogy highlights a similarity with respect to few aspects only. In that case one should either de-emphasize the other aspects, or simply accept the fact that these perceptions are not similar enough for a meaningful analogy to be drawn.

An ideal analogy is almost 'too good' because it means, in a sense, that the two sides of the analogy are essentially the same. An analogy between two really different things typically 'slips' somewhere. P-pullbacks in the proposed category of artificial perceptions are formal tools that can be practically applied for the definition and analysis of analogies that are good but have 'slips'. They formalize a cognitive process of abstraction that extracts the similarity (both in single features and in structure) and separates it from incidental differences.

Another advantage of analogizing with p-pullbacks vs. analogizing with simple arrows is the advantage of symmetry. A p-morphism means that there is a domain that maps into a codomain, and the roles are not symmetric. With a p-pullback there is no restriction about one perception being consistently less blurred or more general than the other. For example, in remark 2 of section 4, it was mentioned that some p-predicate values of the example in that section could be changed. The symmetrical construction of a pullback is more robust to variations of this nature. They should not cause an essentially meaningful analogy to collapse.

There is an additional intuition about analogies and cognitive processes that is formalized by superordinate perceptions and p-pullbacks. In section 5 it was mentioned that the analogy between the camera and the eye may be used to suggest principles of functioning for other optical instruments. Analogizing with superordinate perceptions and p-pullbacks provides basis for a cognitive process where analogies are used to suggest hypotheses about a more general rule or setting. The superordinate perception of an 'image generating system' suggests principles of functional design for other optical instruments: It suggests that such systems are designed by a composition of constituents that refract, adjust focus, etc. This could serve as basis for either analogical designs on one hand, or, on the other hand, for reasoning about optical instruments. When faced with an existing optical instrument, or with a prospective design of such an instrument, a perceptive intelligence that has gone through the perceptual cognitive process that is described by the p-pullback, would perhaps be able to analogize further and look for a refracting modul, for the control hierarchy, etc. Such cognitive processes are typically based on analogies with past perceptual experience, reinforced by generalizing cognitive mechanisms. A variety of analogs is accumulated and unified into a single superordinate analog, providing the additional advantage of avoiding the pitfall of fixation on a single previous analog. The superordinate analog internalizes the idea that focus adjustment, for example, can be achieved in various ways.

Indeed, the more general the setting, the less detailed will be the superordinate perception. If the construction has already 'pulled back' many perceptions of various optical instruments, it may end up with a general observation that the only essential constituent of an optical instrument either refracts or reflects. A combination of generality and specificity could be attained by restriction of the perceptions that are incorporated into the superordinate perception, such as 'perceptions of optical instruments with focus adjustment'.

The projection l-analogies from the superordinate perception to the concrete perceptions that generated the p-pullback can be factorized as in section 6. In that case it would perhaps be more appropriate to entitle the constituents of the intermediate perceptions as 'specific examples' rather than 'metaphors'. There is considerable overlap between terms such as *examples, similes, models, metaphors, analogs* etc. The purpose of this paper is to formally capture the essence of the underlying cognitive processes and to highlight structural similarities between them.

P-products and p-pullbacks are more complex categorical constructs than basic p-morphisms. This seems to capture the intuition that a generalization of two concrete systems into an abstract system requires more cognitive effort than the construction of a simple analogy by one, straight forward, p-morphism. This parallelism between cognitive effort and the complexity of the formal construct was not asserted at the outset. Therefore it provides additional support for the suitability of the proposed schema.

To summarize, analogizing with superordinate perceptions and p-pullbacks has the following advantages:

- It formalizes detection of an essence of similarity, and of structure, if it exists, while offering the compromise that blurs and de-emphasizes differences.
- It formalizes a cognitive process of generalization and abstraction, suggesting the existence of general common principles that underlie the perceptions that are offered as analogs.
- It features a formal symmetry between the two analogs.

The formalism opens the way for programmable procedures that might be capable of detecting similarities between perceptions and deriving analogies. The generation of the p-product, and also the comparison and selection process that yields the p-pullback are quite straight forward. Two relevant issues are worth mentioning:

- A formalism is not an algorithm. There is indeed room for more research on complexity and implementational issues. The categorical construct just offers a basic foundational theoretical standard.
- Following the discussion in section 2.2, the proposed formalism is not meant to be empirically adequate from a human psychological point of view. It is not claimed that human analogy-making actually works by generating a p-product and then a p-pullback, it probably does not. It is often the case in AI that artificial systems typically apply tedious search (as in scanning the entire p-product) where humans would typically conjure specialized effective retrieval methods. Every intelligence does whatever it is good at doing. Computer vs. human chess playing, a landmark success of AI, is probably such an example. It is likely that the internal workings of *Deeper Blue* were not identical to those of Kasparov. However, they did interact in the same game and it is generally accepted that their capabilities may be legitimately compared. Fusing the discussion in section 2.2 with the ideas of this section, the analogy that was suggested between human cognitive processes and artificial systems is roughly sketched in figure 7, where the arrows are *not iso*morphisms.

9 Most Structured Arrows for Analogies

More complex structures in perceptions are lawlike patterns (i.e. subjacencies and congenerics, and also subsumptions and synonyms) that can be observed between *Boolean combinations* of w-elements (and also of connotations). A Boolean combination of connotations can be understood intuitively, as explained in section 3.

Example 6 *In the perception of the eye, the iris* regulates the amount of light AND *is* NOT light sensitive*:*
$\varrho(\mathit{iris}, \text{light amount regulation} \wedge \neg(\text{light sensitivity})) = t$.

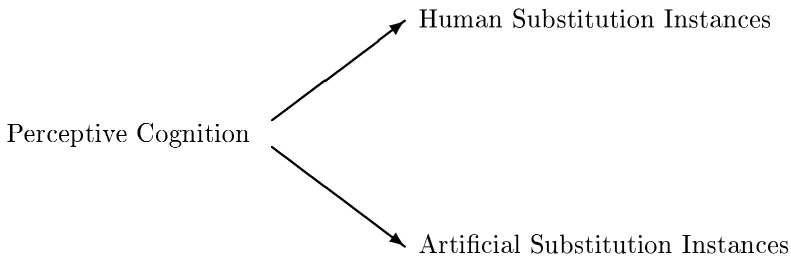

Figure 7: The Categorical Model

An adequate deductive apparatus for the computation of the three-valued p-predicate for such Boolean combinations is provided in [4, p.210-211].

A Boolean combination of w-elements is less intuitive, because it often involves an abstractive conception of environmental elements that do not really exist. However, it can be technically defined in the same way. The same deductive apparatus that was just mentioned for the computation of the three-valued p-predicate for Boolean combinations of connotations can be applied for the computation of the three-valued p-predicate for Boolean combinations of w-elements, mutatis mutandis. Loosely, disjunction of two w-elements is conceived as another w-element that should have the union of the connotations that they have, and should lack the intersection of the connotations that they lack. Conjunction of w-elements is dual: it is conceived as another w-element that should have the intersection of the connotations that they have, and should lack the union of the connotations that they lack. A negated w-element is a w-element that has the connotations that the original w-element lacks, and lacks the connotations that the original w-element has. Negated w-elements do seem less intuitive because the inversion of truth values alienates the simile. However, even in the human context negative examples are acceptable in many domains, and are often considered better than no example at all (e.g. 'See this? - this is exactly what you *do not* want to do'...).

For example, by $\neg(\mathsf{retina})$ one designates an imaginary w-element. It has connotations that are the negations of the connotations of retina, namely it positively features *light amount regulation*, *focus adjustment*, *refraction*, *dynamic change*, and *control of other object*, and it lacks *light sensitivity*. Actually, it is congeneric to a disjunction of iris and $\mathsf{eye\ lens}$:

$$\forall \alpha \in \mathcal{I}_{\mathrm{Eye}} \quad \varrho_{\mathrm{Eye}}(\neg\mathsf{retina}, \alpha) = \varrho_{\mathrm{Eye}}(\mathsf{iris} \vee \mathsf{eyelens}, \alpha)$$

This kind of *Boolean structure*, that consists of lawlike patterns (in the example above: congenerics) also between *Boolean combinations* of constituents, is defined and studied in [4, p.225-227], where p-morphisms that preserve this structure are also defined. This is done for the subcategory $\mathcal{P}rc_{\mathcal{E}}$, and can be easily extended in a straight forward manner to the entire category $\mathcal{P}rc$. Introduction of the Boolean structure into the internal representation $\mathcal{I}$ is systematized by a suitable free endofunctor into the subcategory of *Boolean perceptions* [4, p.228-230]. Introduction of this structure into the perceived environment $\mathcal{E}$ can be dually systematized by a suitable free endofunctor, mutatis mutandis. The import for analogy making, and cognition in general, is that the same perceptual acuity, which is captured by Boolean structure, pervades all cognitive processes: interpretations, communications, mental representations, analogy and metaphor making, as well as other cognitive processes that will be mentioned in section 11 that describes subsequent research.

If the Boolean structure is preserved by an analogical p-morphism, or by a superordinate perception, then the analogy is even more structured, and so are the implied metaphors. Theorem 5 obviously holds also for this extended notion of structure.

Example 7 *In the perceptions of the camera and the eye it can be observed that, for all w-elements* w, ϱ*(*w,light amount regulation $\vee$ focus adjustment*)=*ϱ*(*w,dynamic change*), meaning that* dynamic change *is synonymous to (*light amount regulation $\vee$ focus adjustment*). This is a lawlike pattern of Boolean combination of connotations that is preserved by the l-analogy.*

Some lawlike patterns of Boolean combinations of w-elements in the analogy between the camera and the eye are preserved as well. It can be observed that $\neg$*(film* $\vee$ *camera lens)*$\trianglelefteq$*(outside actions). Namely, an imaginary w-element with all connotations except for* light sensitivity *and* refraction *should have been subjacent to* outside actions*. This subjacency is preserved by the analogy, since* $\neg$*(retina* $\vee$ *cornea)*$\trianglelefteq$*(brain stem reflexes): an imaginary w-element with all connotations except for* light sensitivity *and* refraction *should have been subjacent to* brain stem reflexes*. These patterns are meaningful to the understanding of the functioning of the camera and the eye, and their preservation is, hence, meaningful as well.*

In section 3 it was explained how introduction of classification and organization cognitive processes into the proposed sense perception framework is formalized in the subcategory of *Boolean perceptions* that have sets of connotations that are closed under Boolean operations, namely the $\mathcal{I}$'s are Boolean algebras. A substantial part of [4] was dedicated to the construction of the p-morphisms that map basic neural-sensory-motor perceptions and simple representations into their Boolean closures, producing logical, cognitive, representations. Out of the two canonical free constructions into the Boolean subcategory that formalize generation of meaningful cognitive images of the environment, the one that is more 'perceptually acute', and answers to a criterion of mathematical completeness, is based on the internalization of lawlike patterns (also between Boolean combinations) as described above. This provides a connecting thread between various perceptual-cognitive processes of discerning perceptions. (The association is at all possible because all these processes share the same mathematical, context free, premises from section 3.) Both the generation of a perceptually acute cognitive image of the environment, as well as the generation of highly structured analogies, are based on internalizations of lawlike patterns between (Boolean combinations of) perceptual constituents. There is hence a connection between the capability to provide and fathom insightful analogies and the capability to construct a subtle cognitive image of the environment. The key to both processes is the internalization of as many perceptually observable patterns as possible.

A conclusion with both formal and applicational import is that a single component that should be capable of a certain mental acuity – the detection of lawlike patterns between (Boolean combinations of) perceptual constituents –

is a significant and reusable component for various, seemingly different, intelligent perceptual cognitive processes. A programmed implementation of such a component is described in [13], where they describe a logic-based methodology for analyzing observations that is based on the detection of hidden patterns in the data. Combinations of such patterns are used for developing general classification procedures. We propose that similar methodologies can be applied by an intelligent artifact to embodied ontologies, and that the classifications be used for more cognitive activities. With a neat distinction among subjective perceptions, external environments, and internal representations, the mathematical categorical toolkit of the theory of artificial perceptions provides basis for employing the Boolean methodology for a broader spectrum of cognitive activities.

10 Cognitive Science Revisited

In section 2 it was suggested that one should return to theories of human cognition in order to examine the results of this study. This dialogue between the artificial-formal and the natural-human contexts is resumed to tie the ends of our intuitions. Indeed, ideas such as:

- Metaphors are implied by analogies...
- One may generalize from analogs...
- Subtle, acute perceptions provide insightful analogies and metaphors...
- Subtle, acute perceptions classify and organize...

...Are quite intuitive and not new. The mathematical infrastructure that is provided by the definition and study of the category and the theory of artificial perceptions has enabled us to extend, to systematize, and to formalize these intuitive pre-theoretical conceptions in a parallel, categorical, setting. By employing well-developed tools of mathematics, it is thus possible to model complex perceptual cognitive phenomena that could be grasped only loosely by verbal descriptions. The mathematical categorical framework models them in a precise, testable and applicable form, augmenting the evidence from [5, 4] that mathematical categorization of artificial perceptions can be useful to AI.

There is an additional 'intuition about intuitions' that is extended by the parallelism. A prudent study of analogies should be conducted with a reservation. The fruitfulness of analogies depends on whether any testable consequences can be deduced from them, which is likely to depend on whether the resemblance is fundamental or superficial. However, when analogies are employed, it is not always possible to show that the resemblances noted bear relevantly on the points to be established, whereas the differences are irrelevant. It is not always possible to support the analogy with independently established considerations, and it is not always the case that if things are alike in some ways, they will be alike in others, hence analogies should not be 'carried too far'.

With all these well known limitations, researchers still often view analogizing as a fundamental paradigm of cognitive capabilities, analogy following and analogy making being hallmarks of intelligence tests. Fauconnier [21], for example, examines the mappings that link mental spaces as a central component of meaning construction. He argues that the same principles operate at the highest levels of scientific, artistic, and literary thought, and at the lower level of elementary understanding. He counts analogical mappings as a key cognitive operation, along with conceptual integration and blending, discourse management, induction, and recursion. Another example is the analysis of metaphor as a special conveyer of meaning, and as a rich cognitive device, that is carried by Ross [53]. He links the study of metaphor processing to wider issues in cognitive science.

The proposed theory of artificial perceptions may offer its parallel clue to the paradigmatic significance of analogy-making. Arrows are, indeed, the most fundamental component of categorical constructions. A competence in fabricating arrows is thus indicative of a potential capability to erect all other categorical constructions as well. In the proposed categorical formalism many cognitive processes are systematized by p-morphisms. It was already argued in section 5 that a p-morphism can be regarded as a generalized notion of an analogy. It follows that an intelligent perceptive artifact that is competent at analogizing, holds a fundamental key to other categorical constructions of the theory, namely to other perceptual cognitive processes that are formalized by this setting. This parallelism between the significance of arrows in the categorical setting, and the significance of analogies in human cognition, provides an additional supporting argument that the proposed theory provides a fruitful and effective formalization of intuitions about perceptual cognitive processes, namely that the analogs *drawn here* (between the artificial-formal and the natural-human contexts) are not superficial, and that they bear relevantly on perceptual cognitive processes.

11 Summary and Subsequent Work

It is often the case in AI and in cognitive science, and also in category theory and in logic, that one applies a methodology while studying it. In this work an analogy has been drawn between perceptual cognitive processes of analogy and category theoretical constructs. It was shown how quite a few properties and structural features of cognitive processes are preserved and systematized in the categorical setting. Using the terminology of Magnan and Reyes [48], the consequences of this analogy provide basis for 'blueprints' that can be followed in an artificial perceptual cognitive context.

Subsequent research, some of which builds on the constructs that were introduced in this paper, is concerned with further cognitive processes that can be captured by the same mathematical categorical framework. Cognitively conceived creative design processes are formalized by natural transformations of environments in [3]. This offers additional parallelisms concerning the connection between reasoning processes, design processes, and analogies. Another research direction [2] is concerned with an extension of the formalism to derive from within, in a categorical manner, emotive reactions that may be triggered by perception (a shift in the focus of attention is an example). This provides premises for the incorporation of behavior, reactions, conflicting reactions, prioritizing, and eventually also affective elements, into the cognitive processes that are captured by this framework. This should offer a formalization of the interfusion of the sensible and the sensitive aspects of embodied artificial intelligence. Since all the perceptual cognitive processes are based on the same premises, they can form an integrated schema of affective artificial cognition. Integration, in the categorical sense, is the eventual construction of a commutative diagram that interweaves the processes.

This study suggests a 'theoretical standard' against which analogies and related perceptual cognitive processes can be generated and analyzed in an artificial intelligent context. It does not provide a programmed or an algorithmic implementation. However, the analysis in terms of the relatively small number of primitives of the proposed category (w-element, connotation, perception predicate), supported by category theoretical constructions, predicts the possibility of tidily structured implementations with a reduced component set. Context free categorical procedures, such as p-morphisms, seem to be reusable for different cognitive processes. It was also shown how a context free 'mental' component that detects lawlike Boolean patterns of perceptual constituents could be reusable for various artificial cognitive activities.

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